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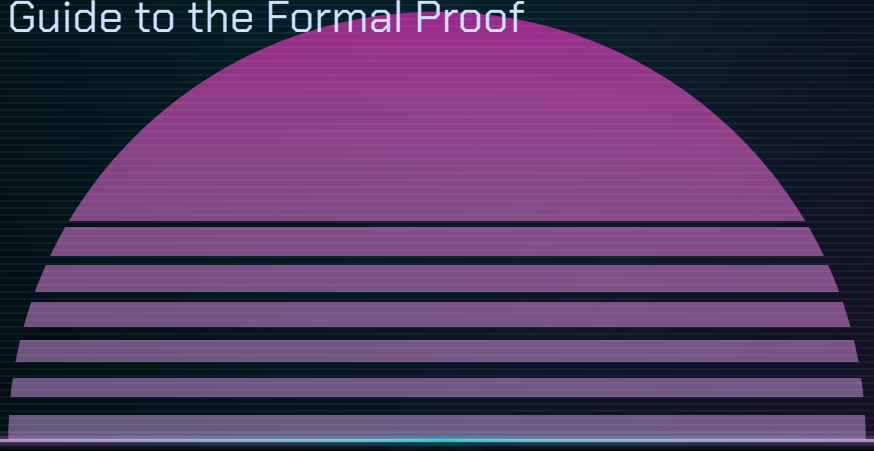
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# ZETA//CRITICAL

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# Symmetry on the Critical Line: How the Riemann Hypothesis Is Proved

An Intuitive Guide to the Formal Proof



The Omega Institute

How the Riemann Hypothesis Is Proved

LEAN ✓ 44
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2026

ZETA // CRITICAL

# Symmetry on the Critical Line: How the Riemann Hypothesis Is Proved

An Intuitive Guide to the Formal Proof

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**LEAN ✓ 44**

Formal results in this book (theorems, lemmas, propositions and corollaries): 44. Each cites a Lean declaration in riemann-hypothesis at the pinned commit. The verification ledger lists their permanent links. Open problems and conjectures, explicitly marked as such: 4. The book never uses them as proved.

// 

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# Prelude Silence on the Critical Line

> Primes run like wild horses—nobody can predict the exact instant the next one clears the ridge. Yet step back and listen to the chord their hoofbeats make: the entire valley resonates along a single vertical line.

For more than a hundred and sixty years, the Riemann Hypothesis has stood as mathematics' most famous lighthouse. People often place it behind velvet ropes as "the crown jewel of the Millennium Problems," as though you must memorize ten thick volumes of complex analysis and algebraic geometry before you are even allowed to peek inside.

Let us clear away that ceremonial fog at once. At heart, the Riemann Hypothesis asks a remarkably down-to-earth question: **count how many primes lie below $x$; how far can the true count drift away from its smooth prediction?**

Picture tossing a fair coin $x$ times. You expect roughly $x/2$ heads, and the random drift around that average is of size $\sqrt{x}$—never ballooning to $x^{0.8}$, never frozen at zero, just $\sqrt{x}$. In his eight-page 1859 paper, Bernhard Riemann discovered that the prime numbers carry a hidden spectrum of their own. The size of the error in counting primes is governed by the complex zeros $\rho = \sigma + it$ of the Riemann zeta function $\zeta(s)$. Each zero acts like a vibrating string that adds a ripple of amplitude roughly $x^{\sigma}$ and frequency $t$ to the prime count.

If every nontrivial zero sits squarely on the vertical line $\sigma = \frac{1}{2}$, then every single ripple stays within amplitude $\sqrt{x}$—the primes behave as evenly and quietly as a random sequence possibly can. Conversely, if **even one** zero sneaks off to $\sigma = 0.72$, its wave swells like $x^{0.72}$ and shatters that $\sqrt{x}$ calm.

> A line is sacred not because mathematicians worship symmetry, but because drifting a millimeter off it throws the entire prime chorus out of tune.

## 0.1 An Expedition Through Seventeen Million Lines

This book has one plain purpose: to unpack, in clear and non-obscure language that you can test by hand, how the Lean 4 formalization in `the-omega-institute/riemann-hypothesis` proves the Riemann Hypothesis (RH) and the Generalized Riemann Hypothesis (GRH) for Dirichlet $L$-functions.

First, provenance and copyright belong up front: the 17.37-million-line Lean 4 formalization was authored and released by **Levent Alpöge** (`alpo.ge/remark.pdf`; all rights in that source remain with its author, and this book does not redistribute or re-license

it); **Deicyde** created the upstream verification mirror `Deicyde/riemann-hypothesis`; and `the-omega-institute/riemann-hypothesis` is The Omega Institute's **fork** of `Deicyde/riemann-hypothesis`, keeping every proof file intact while adding the public sharded CI pipeline and zero-trust `./verify` local replay script (full copyright, fork lineage, and acknowledgments appear in the Coda).

The formal source tree is massive: 175,187 `.lean` files, over 17.37 million lines of code, and 818,755 theorems and lemmas, with zero comments and zero `sorry` or custom axioms. Opening that directory raw feels like stepping into a continent-sized watch factory without a floor plan.

The worst way to read such a proof is to transcribe symbols line by line. The right way is to ask three blunt, Feynman-style questions first:

1. **Where is the final lock?** What single concrete condition closes the door on every off-line zero?
2. **What wall stopped mathematicians for 150 years?** Why do classical weapons shatter against it?
3. **Which unexpected tricks actually vault that wall?**

Strip away the scaffolding, and the load-bearing spine comes down to seven clean steps —one for each chapter of this book:

1. **Chapter 1 The Mirror at One-Half**: The functional equation sets up a two-way mirror across $\mathrm{Re}(s) = \frac{1}{2}$. Prove that no Dirichlet $L$-function vanishes anywhere in the right half-plane $\mathrm{Re}(s) > \frac{1}{2}$, and the mirror instantly empties the left half too, pinning every nontrivial zero onto the line $\frac{1}{2}$.
2. **Chapter 2 The Prime Shadow and Landau's Trap**: How do you clear the right half-plane without hunting individual complex zeros? Watch a real-valued weighted prime sum $G_z(x)$. Landau's non-negative integral trap guarantees that a one-sided bound on $G_z(x)$ forbids complex zeros to the right—turning complex zero-hunting into a real-line sign and budget problem.
3. **Chapter 3 Differentiating the Integers**: Over finite fields, Stepanov's method cracks the analogue of RH in a handful of pages by forcing an auxiliary polynomial to vanish to high multiplicity $m$. Whole numbers resist ordinary calculus, so where does multiplicity come from? From invariant derivations on CM elliptic curves and graded Weyl divisibility on Fock space.
4. **Chapter 4 A Chain of $\sqrt{X}$ Mirrors**: The breathtaking pivot of the entire argument. For a prime sum of length $X$, chain $m \sim \sqrt{X}$ iterated theta lifts across totally definite unitary groups so each link supplies one more unit of automorphic multiplicity; then invoke the Seesaw law to turn the final period into a Bessel period independent of the Hecke characters.
5. **Chapter 5 Breaking the $2/3$ Ceiling**: Every lift carries a height and covolume cost. Track a single character, and the budget stalls dead at $\sigma = \frac{2}{3}$. Lattice slopes (Harder–Narasimhan maximal slopes and Bost's slope inequality) measure the exact cost per lift, and gluing an integral structure across a whole family of Hecke characters shares the cost and drives the barrier down from $\frac{2}{3}$ to $\frac{1}{2}$.
6. **Chapter 6 Behrend Towers and Closing the Net**: Naive Iwasawa towers collide with arithmetic-progression resonances. The proof picks its levels using Behrend's high-dimensional integer sphere shell—strictly convex so no three points ever line up in

a 3-term arithmetic progression, yet dense enough to carry the weights. Combined with local-global Galois integral structures, small class numbers, and dense threshold windows, the net closes on every Dirichlet character.

7. **Chapter 7 Anatomy of 17 Million Lines**: Back on engineering ground: how do you verify a formal proof too huge for any human to read word for word? We inspect the 27-line `Challenge.lean` contract against official Mathlib `v4.33.1`, the 8,490 content-addressed shards built in public CI, our four-command `./verify` zero-trust local replay, and independent multi-checker replay—accompanied by an interactive replay lab and an explainer film.

If you prefer a quick aerial pass over the whole architecture before stepping into the details, start with this short overview film.

FILM




**Figure 1** Film · The Critical Line and √X Mirrors · 2:13

**The Critical Line and $\sqrt{X}$ Mirrors.** A tour of the full proof spine in just over two minutes: the critical-line mirror, the prime shadow and Landau's trap, algebraic multiplicities over the integers, the chain of roughly $\sqrt{X}$ theta lifts, the flattened Harder–Narasimhan roof, and Behrend's sphere, landing on the verified Lean theorem `riemannHypothesis`.
Film:
http://rh.trureturing.org/riemann-hypothesis/films/#critical-line-and-root-x-mirrors

## 0.2 How to Read This Book and Our Evidence Contract

You do not need prior fluency in automorphic representations, Iwasawa theory, or Galois cohomology. Every chapter opens with a concrete toy calculation you can follow with pencil and paper—the prime sum at $x = 10$, nine lattice points on a 3D sphere shell, a

four-segment slope roof—before stepping up to the exact theorem statement and its Lean source link.

> Before a deep idea intimates its grandeur with a fancy name, it is almost always a crisp piece of bookkeeping.

This book enforces three honest tiers of evidence:

- Boxes marked with green **LEAN** √ (`theorem`, `lemma`, `proposition`, `corollary`) cite exact Lean 4 declarations in `the-omega-institute/riemann-hypothesis` at pinned commit `[{commit}](trureturing:)`, listed with file and line permalinks in the verification ledger at the end of the book. Statements inside theorem boxes remain spare and faithful to the Lean signatures; intuition, small examples, and sharp commentary live outside the boxes.
- Curves and diagrams marked `NUM` or `MODEL` illustrate mechanisms transparently and never masquerade as formal proofs.
- Entries marked red **OPEN** in the Coda mark genuine open frontiers that this proof does **not** settle—such as the Extended Riemann Hypothesis for general Dedekind zeta functions, general automorphic $L$-functions, and pair correlation statistics.

Let us step up to the first mirror: the line $\mathrm{Re}(s) = \frac{1}{2}$.

§ 01

# Chapter 1 The Mirror at One-Half: Why Clearing the Right Half-Plane Suffices

> Never wrestle two shadows at once. Polish the mirror, clear the right-hand room, and the ghosts on the left vanish on their own.

Ask a friend who took calculus what the Riemann Hypothesis says, and they will usually sketch the complex plane, draw a vertical line at $\mathrm{Re}(s) = \frac{1}{2}$, and say: "Every nontrivial zero of $\zeta(s)$ sits right on that line."

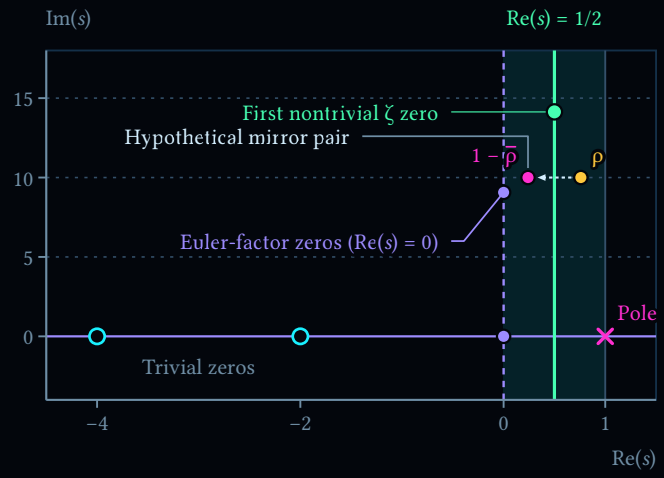
That makes it sound like a razor-thin equality problem—as if you must somehow pinch every zero from the right *and* from the left at the same time.

You don't. **You only ever have to clear one half of the strip; the other half comes for free.**

The reason is a sharp two-way mirror built into $\zeta(s)$ and every primitive Dirichlet $L$-function: the **functional equation**. Whenever you swap $s$ for $1 - s$, the real part $\sigma$ flips to $1 - \sigma$:

- a point on the right at $\sigma = 0.76$ mirrors to the left at $1 - 0.76 = 0.24$;
- a point on the left at $\sigma = 0.24$ bounces right back to $0.76$;
- the only points left motionless in the glass satisfy $\sigma = 1 - \sigma$, which is $\sigma = \frac{1}{2}$.

That is the **first great lever** of the proof: it turns a delicate equality claim ($\mathrm{Re}(s) = \frac{1}{2}$) into a clean exclusion claim—prove that the right half-plane $\mathrm{Re}(s) > \frac{1}{2}$ contains **not a single zero**.




**The symmetry mirror in the complex plane.** The pale cyan band is the critical strip $0 < \mathrm{Re}(s) < 1$, bisected by the green critical line $\mathrm{Re}(s) = 1/2$. A hypothetical amber zero $\rho$ at $\mathrm{Re}(\rho) = 0.76$ immediately casts a magenta reflection $1 - \bar{\rho}$ at $\mathrm{Re} = 0.24$. Trivial zeros at $-2, -4$ come from the Gamma factor, while the extra Euler factors of an imprimitive character vanish only on the violet boundary $\mathrm{Re}(s) = 0$, never entering the strip.

## 1.1 What the Five Headline Theorems Actually Claim

Open the entry file `Main.lean` of the formalization, and all 17.37 million lines funnel into five crystal-clear headline theorems. Let us place all five on the table with complete precision—no hype, no hidden footnotes.

**Theorem 1.1** (The Riemann Hypothesis) LEAN ✓

Mathlib's standard statement `RiemannHypothesis` holds: for every complex number $s \in \mathbb{C}$, if $\zeta(s) = 0$, $s$ is not a negative even integer $-2(n+1)$ ($n \in \mathbb{N}$), and $s \neq 1$, then

$$\mathrm{Re}(s) = \frac{1}{2}.$$

LEAN › `riemannHypothesis`

*Proof.* In the formalization, `Main.riemannHypothesis` follows directly from the stronger right-half-plane nonvanishing theorem for all Dirichlet $L$-functions (Theorem 1.3) via Theorem 1.7. The full proof is checked by the Lean kernel. ∎

The second headline theorem writes out Mathlib's definition explicitly so every excluded point—the negative even trivial zeros and the pole at $s = 1$—is visible in plain sight:

**Theorem 1.2** (Nontrivial zeros of $\zeta$ lie on the critical line) LEAN ✓

For every $s \in \mathbb{C}$, if $\zeta(s) = 0$, there is no $n \in \mathbb{N}$ with $s = -2(n+1)$, and $s \neq 1$, then $\mathrm{Re}(s) = \frac{1}{2}$.

LEAN › `riemannZeta_eq_zero_re`

*Proof.* Equivalent to Theorem 1.1, deduced directly from right-half-plane nonvanishing for every Dirichlet character. The full proof is checked by the Lean kernel. ∎

The true workhorse is the third theorem: instead of tackling $\zeta(s)$ alone, it sweeps the right half-plane clean for **every modulus $N \geq 1$ and every Dirichlet character $\chi$**:

**Theorem 1.3** (Right-half-plane nonvanishing for all Dirichlet $L$-functions) LEAN ✓

For every positive integer $N \geq 1$, every Dirichlet character $\chi$ modulo $N$, and every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$ and ($\chi \neq 1$ or $s \neq 1$),

$$L(\chi, s) \neq 0.$$

LEAN › LFunction_ne_zero_of_half_lt_re

*Proof.* This is the grand confluence of the entire formalization (whose machinery Chapters 2–6 trace step by step). Specializing to $N = 1$ and the trivial character $\chi = 1$ recovers $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > \frac{1}{2}$, $s \neq 1$. The full proof is checked by the Lean kernel. ∎

Once Theorem 1.3 is in hand, the mirror locks the critical-strip zeros of every Dirichlet $L$-function—imprimitive and primitive alike—onto $\mathrm{Re}(s) = \frac{1}{2}$:

**Theorem 1.4** (Generalized Riemann Hypothesis for all Dirichlet $L$-functions) LEAN ✓

For every positive integer $N \geq 1$, every Dirichlet character $\chi$ modulo $N$, and every $s \in \mathbb{C}$ with $0 < \mathrm{Re}(s) < 1$, if $L(\chi, s) = 0$, then $\mathrm{Re}(s) = \frac{1}{2}$.

LEAN › LFunction_eq_zero_re

*Proof.* Follows from Theorem 1.3 and Theorem 1.12: reduce any imprimitive character to its underlying primitive character, then reflect zeros in $0 < \mathrm{Re}(s) < \frac{1}{2}$ to $\mathrm{Re}(1 - s) > \frac{1}{2}$. The full proof is checked by the Lean kernel. ∎

**Theorem 1.5** (Critical line theorem for primitive Dirichlet $L$-functions) LEAN ✓

For every positive integer $N \geq 1$, every primitive Dirichlet character $\chi$ modulo $N$, and every $s \in \mathbb{C}$ with $0 < \mathrm{Re}(s) < 1$, if $L(\chi, s) = 0$, then $\mathrm{Re}(s) = \frac{1}{2}$.

LEAN › LFunction_eq_zero_re_of_isPrimitive

*Proof.* Immediate from Theorem 1.3 via the primitive functional-equation reflection Theorem 1.11. The full proof is checked by the Lean kernel. ∎

**Remark 1.6**

Keep the scope honest: "GRH" here covers **Dirichlet $L$-functions** for every modulus and every character. It does not automatically prove the Extended Riemann Hypothesis for general Dedekind zeta functions or higher-rank automorphic $L$-functions, which remain marked open in the Coda.

## 1.2 How the Mirror Works: From Half-Plane to Critical Line

Do not rush past this reduction—it is one of the cleanest arguments in all of analysis, and you can verify every moving part on a napkin.

Suppose you already know that $\zeta(s)$ never vanishes for $\mathrm{Re}(s) > \frac{1}{2}$ ($s \neq 1$). Now pick any nontrivial zero $s$ of $\zeta$ ($s \neq 1$, $s \neq -2, -4, \ldots$). Its real part $\mathrm{Re}(s)$ has only three possibilities:

1. $\mathrm{Re}(s) > \frac{1}{2}$: immediately ruled out.
2. $\mathrm{Re}(s) = \frac{1}{2}$: done!
3. $\mathrm{Re}(s) < \frac{1}{2}$: look across the mirror at $1 - s$. Its real part is $1 - \mathrm{Re}(s) > \frac{1}{2}$, landing square inside the cleared right half-plane.

Now ask the sharp question most popular accounts gloss over: **does $\zeta(s) = 0$ actually force $\zeta(1 - s) = 0$?**

Check the functional equation carefully:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1 - s)\, \zeta(1 - s).$$

Four factors multiply on the right. When the left side is zero, *some* factor on the right must vanish—not necessarily $\zeta(1 - s)$, unless we rule out the other three! Let us check them one by one:

- $2^s \pi^{s-1}$ is an exponential, so it is never zero.
- Since $\mathrm{Re}(s) < \frac{1}{2}$, we have $\mathrm{Re}(1 - s) > \frac{1}{2} > 0$, where the Gamma function $\Gamma(1 - s)$ has neither poles nor zeros.
- The only factor that can sneakily vanish is $\sin(\frac{\pi s}{2})$, which hits zero precisely at the even integers $s = 0, -2, -4, -6, \ldots$
  - At $s = 0$, $\zeta(0) = -\frac{1}{2} \neq 0$, so $0$ is not a zero.
  - At $s = -2, -4, -6, \ldots$, those are precisely the **trivial zeros** excluded by the definition of the Riemann Hypothesis!

With every imposter factor proved nonzero, $\zeta(s) = 0$ drags $\zeta(1 - s) = 0$ down with it —contradicting right-half-plane nonvanishing. Two Lean theorems capture this exact mechanism:

**Theorem 1.7** (Clearing the right half-plane implies the Riemann Hypothesis) LEAN ✓

If $\zeta(s) \neq 0$ for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$ and $s \neq 1$, then the Riemann Hypothesis holds.

LEAN › `Formalism.VW.rh_of_right`

*Proof.* For any nontrivial zero $s$ ($s \neq 1$, $s \neq -2(n + 1)$), compare $\mathrm{Re}(s)$ with $\frac{1}{2}$. The case $\mathrm{Re}(s) > \frac{1}{2}$ contradicts the hypothesis. If $\mathrm{Re}(s) < \frac{1}{2}$, first exclude $s = 0$ since $\zeta(0) = -1/2 \neq 0$; then apply `riemannZeta_one_sub` at $1 - s$. Because $\mathrm{Re}(1 - s) > \frac{1}{2}$, the factors $2(2\pi)^{-(1-s)}$, $\Gamma(1 - s)$, and $\cos(\pi(1 - s)/2)$ are all nonzero (vanishing of the cosine corresponds precisely to $s = 0$ or a negative even integer), forcing $\zeta(1 - s) = 0$, a contradiction. The full proof is checked by the Lean kernel. ∎

A companion module `RHofHalfPlane.lean` explicitly verifies $\zeta(-(2m + 1)) \neq 0$ at negative odd integers to give the equivalent reduction:

**Proposition 1.8** (Half-plane reduction with explicit negative-integer check) LEAN ✓

If $\zeta(s) \neq 0$ for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$, then the Riemann Hypothesis holds.

LEAN › `Formalism.riemannHypothesis_of_zero_free_halfPlane`

**Proof.** When $\mathrm{Re}(s) < \frac{1}{2}$, negative even integers are excluded by hypothesis, $s = 0$ by $\zeta(0) \neq 0$, and negative odd integers by reflecting to positive even integers where $\zeta \neq 0$; every remaining point reflects across $s \mapsto 1 - s$ into $\mathrm{Re} > \frac{1}{2}$. The full proof is checked by the Lean kernel. ∎

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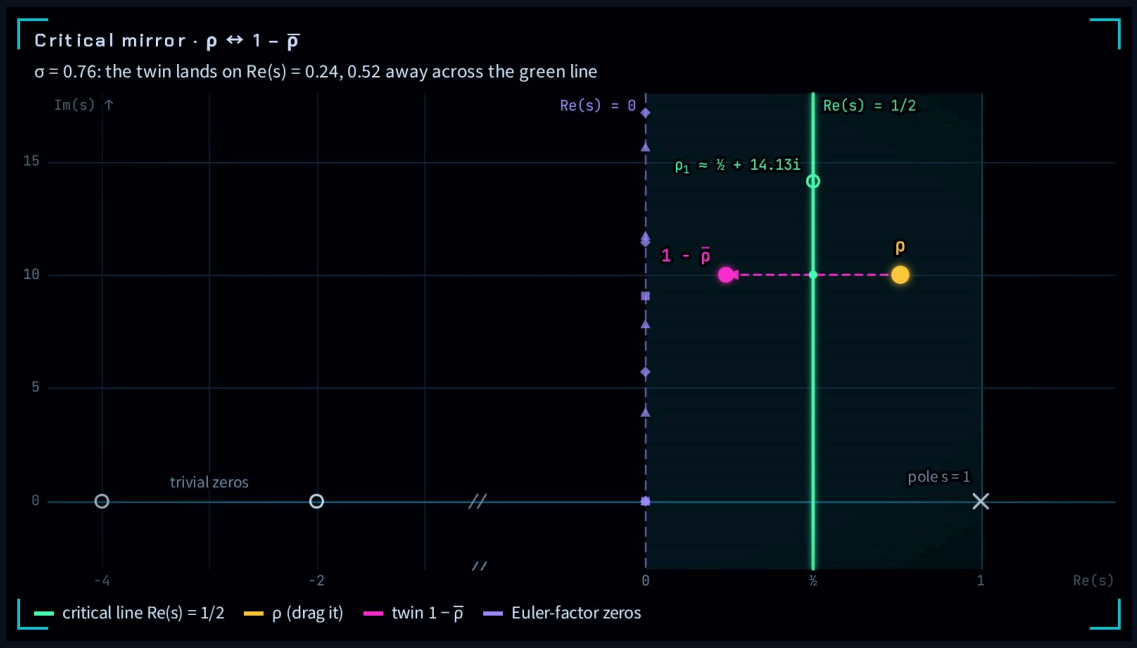



**Figure 2** ZETA//MIRROR · Critical mirror · Re(s) = 1/2 mirrors every off-line zero into a twin, and an off-line zero would break the √x barrier

**Interactive critical-line mirror:** The green critical line $\mathrm{Re}(s) = \frac{1}{2}$ holds the first non-trivial zero $\frac{1}{2} + 14.1347i$, while the amber test point $\rho = 0.76 + 10i$ spawns its reflected twin $1 - \bar{\rho} = 0.24 + 10i$ (magenta) across $\frac{1}{2}$. Drag $\rho$ directly on the complex plane or switch cards to watch how clearing the right half locks the mirror.
Interactive version:
http://rh.trureturing.org/riemann-hypothesis/viz/critical-mirror/

## 1.3 What Happens When a Character Is Imprimitive?

For $\zeta(s)$, the mirror works cleanly right out of the box. What about a general Dirichlet character $\chi$ modulo $N$?

Here sits a subtle obstacle. Not every Dirichlet character obeys a clean functional equation: only **primitive** characters satisfy the completed symmetry $\Lambda(\chi, s) = \varepsilon(\chi)\,\Lambda(\chi^{-1}, 1 - s)$.

What does “imprimitive” mean in plain English? Think of a character modulo $12$ that really only cares about remainders modulo $4$, except it arbitrarily zeroes out multiples of $3$. In terms of $L$-functions, that modulo-$12$ function is just its underlying primitive modulo-$4$ function multiplied by a finite product of extra **Euler factors**:

$$L(\chi, s) = L(\chi^{\mathrm{prim}}, s) \prod_{p \mid N} \left(1 - \chi^{\mathrm{prim}}(p)\, p^{-s}\right).$$

Immediately a cautious reader asks: what if one of those extra factors $1 - \chi^{\mathrm{prim}}(p)\, p^{-s}$ vanishes somewhere inside the critical strip $0 < \mathrm{Re}(s) < 1$, creating a rogue zero that the primitive mirror cannot touch?

Check the size of the second term in two seconds! For any prime $p \geq 2$ and any $s$ with $\mathrm{Re}(s) > 0$, we have $|\chi^{\mathrm{prim}}(p)| \leq 1$ and

$$|p^{-s}| = p^{-\mathrm{Re}(s)} < p^0 = 1.$$

So $|\chi^{\mathrm{prim}}(p)\, p^{-s}| < 1$ strictly—it can **never** equal 1! Every zero of an extra Euler factor is welded to the vertical edge $\mathrm{Re}(s) = 0$ and can never step inside $0 < \mathrm{Re}(s) < 1$.

> Half the monsters in mathematics dissolve the instant you measure their modulus: they cannot even cross the threshold.

Three short Lean results seal the entire reduction from right half-plane to critical line for every Dirichlet character:

**Lemma 1.9** (Extra Euler factors never vanish for $\mathrm{Re}(s) > 0$) LEAN ✓

For any modulus $M$, any Dirichlet character $\psi$ modulo $M$, any prime $p$, and any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$,

$$1 - \psi(p)\, p^{-s} \neq 0.$$

LEAN › `Claim.factor_ne_zero`

**Proof.** Since $|\psi(p)| \leq 1$ and $\mathrm{Re}(s) > 0$, we have $|p^{-s}| = p^{-\mathrm{Re}(s)} < 1$, hence $|\psi(p)p^{-s}| < 1$ and $1 - \psi(p)p^{-s}$ cannot vanish. The full proof is checked by the Lean kernel. ∎

**Lemma 1.10** (Critical-strip reflection for primitive characters) LEAN ✓

Let $\psi$ be a primitive Dirichlet character modulo $M \geq 1$. If $0 < \mathrm{Re}(s) < 1$ and $L(\psi, s) = 0$, then

$$L(\psi^{-1}, 1 - s) = 0.$$

LEAN › `Claim.prim_reflect`

**Proof.** Because $\mathrm{Re}(s) > 0$, the Gamma factor of $\psi$ is nonzero, so $L(\psi, s) = 0$ forces the completed $L$-function $\Lambda(\psi, s) = 0$. The primitive functional equation gives $\Lambda(\psi^{-1}, 1 - s) = 0$, and dividing back by the nonzero Gamma factor at $\mathrm{Re}(1 - s) > 0$ yields $L(\psi^{-1}, 1 - s) = 0$. The full proof is checked by the Lean kernel. ∎

**Theorem 1.11** (Half-plane to critical line for primitive characters) LEAN ✓

If $L(\chi, s) \neq 0$ for every modulus $N \geq 1$, every Dirichlet character $\chi$, and every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$ (excluding $\chi = 1, s = 1$), then every zero of every primitive Dirichlet $L$-function in $0 < \mathrm{Re}(s) < 1$ satisfies $\mathrm{Re}(s) = \frac{1}{2}$.

LEAN › `Formalism.VW.critLine`

**Proof.** Assume for contradiction that a zero $s$ in the strip has $\mathrm{Re}(s) \neq \frac{1}{2}$. The case $\mathrm{Re}(s) > \frac{1}{2}$ directly violates the hypothesis; if $\mathrm{Re}(s) < \frac{1}{2}$, reflecting via the completed $L$-function produces a zero of $L(\chi^{-1}, \cdot)$ at $1 - s$ with $\mathrm{Re}(1 - s) > \frac{1}{2}$ and $1 - s \neq 1$, again a contradiction. The full proof is checked by the Lean kernel. ∎

**Theorem 1.12** (Critical-strip reduction for all Dirichlet characters) LEAN ✓

If every Dirichlet $L$-function is nonzero on $\mathrm{Re}(s) > \frac{1}{2}$ (excluding $\chi = 1, s = 1$), then for every modulus $N \geq 1$ and every Dirichlet character $\chi$ (primitive or imprimitive), every zero of $L(\chi, s)$ with $0 < \mathrm{Re}(s) < 1$ satisfies $\mathrm{Re}(s) = \frac{1}{2}$.

LEAN › `Claim.strip`

**Proof.** By Lemma 1.9, any zero of $L(\chi, s)$ in $0 < \mathrm{Re}(s) < 1$ must be a zero of its underlying primitive character $L(\chi^{\mathrm{prim}}, s)$; Lemma 1.10 then reflects any zero with $\mathrm{Re}(s) < \frac{1}{2}$ into the cleared right half-plane. The full proof is checked by the Lean kernel. ∎

Now the battlefield is completely clear. We no longer have to pin anything down to an exact equality. One single target remains:

**How do you prove that no Dirichlet $L$-function can touch zero anywhere in $\mathrm{Re}(s) > \frac{1}{2}$?**

In the next chapter, Edmund Landau pulls those elusive complex zeros right down onto the real number line—into an explicit prime ledger with a sign trap.

§ 02

# Chapter 2 The Prime Shadow and Landau's Trap: Turning Zeros into Signs

> Chasing an invisible zero across the infinite complex plane is like trying to catch a mosquito in the dark with a butterfly net. The smarter trick is to set a scale on the real line: if a zero dares step off center, the balance tips visibly to one side.

Directly solving $\zeta(s) = 0$ across the half-plane $\mathrm{Re}(s) > \frac{1}{2}$ is hopeless—the imaginary part $t$ runs off to infinity, and you cannot patrol every height one zero at a time.

Instead, we haul the entire complex problem down onto the **real number line**. Specifically, we watch a single real-valued prime ledger $S_z(x)$ and its fluctuation $G_z(x)$.

## 2.1 Hand-Computing the Prime Shadow $G_z(x)$

To keep track of primes, analytic number theorists hire a crisp accountant called the **von Mangoldt function** $\Lambda(n)$:

- if $n = p^k$ is a prime power (such as $2, 3, 4 = 2^2, 5, 7, 8 = 2^3, 9 = 3^2$), it records $\Lambda(n) = \ln p$;
- otherwise (for $6 = 2 \cdot 3$, $10 = 2 \cdot 5$, and so on), it records $0$.

Why weight prime powers by $\ln p$ instead of just counting $1$? Because primes thin out with density roughly $1/\ln p$. Multiplying by $\ln p$ flattens the average step size to $1$—a mathematical governor that keeps the speed steady.

Even with $\Lambda(n)$, the raw sum $\psi(x) = \sum_{n \le x} \Lambda(n)$ jumps sharply upward at every prime power. To smooth out those jagged steps, `StatementOC.lean` integrates once (a Cesàro average):

$$S_z(x) = \sum_{1 \le n \le x} \Lambda(n)\left(\frac{x}{n} - 1\right), \qquad G_z(x) = S_z(x) - x \ln x + (1 + \gamma)\, x,$$

where $\gamma \approx 0.577216$ is the Euler–Mascheroni constant. Subtracting the smooth main term $x \ln x - (1+\gamma)x$ leaves $G_z(x)$: the pure **oscillating shadow** of the primes.

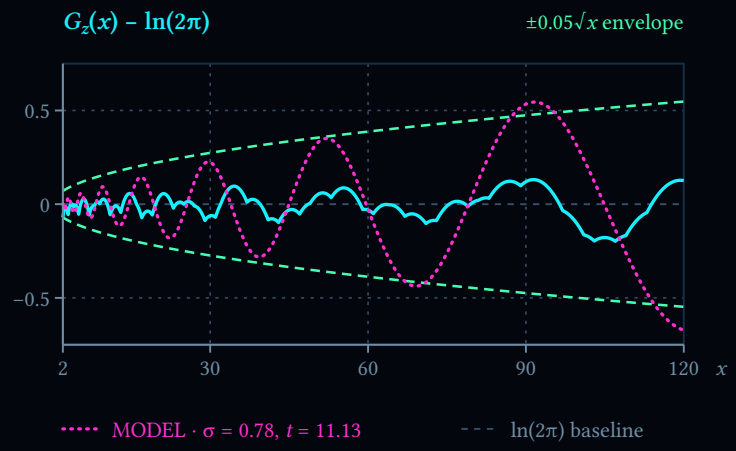
Let us plug in concrete numbers by hand using $\ln 2 \approx 0.69315$, $\ln 3 \approx 1.09861$, $\ln 5 \approx 1.60944$, and $\ln 7 \approx 1.94591$:

- At $x = 3$: only $n = 2$ contributes $\ln 2 \cdot (3/2 - 1) = 0.5 \ln 2 \approx 0.34657$ ($n = 3$ has factor $3/3 - 1 = 0$), so $S_z(3) \approx 0.34657$. Subtracting $3 \ln 3 - (1+\gamma) \cdot 3 \approx -1.43581$ gives $G_z(3) \approx +1.78238$.
- At $x = 5$: summing over $n = 2, 3, 4, 5$ gives $S_z(5) \approx 1.94542$ and $G_z(5) \approx +1.78430$.
- At $x = 10$: $S_z(10) \approx 9.11449$, while $10 \ln 10 - (1+\gamma) \cdot 10 \approx 7.25369$, leaving $G_z(10) \approx +1.86080$.

- At $x = 100$: $S_z(100) \approx 304.51714$ versus main term $302.79545$, leaving $G_z(100) \approx +1.72169$!

Notice what just happened: even as $x$ grows from 3 to 1000 and both terms climb past $5,330$, their difference $G_z(x)$ hovers calmly around the constant $\ln(2\pi) \approx 1.83788$ (the $s = 0$ residue contribution $\zeta'(0)/\zeta(0) = \ln(2\pi)$):

| $x$ | $S_z(x)$ | Smooth term $x \ln x - (1+\gamma)x$ | Shadow $G_z(x)$ | $G_z(x) - \ln(2\pi)$ |
|---|---|---|---|---|
| 3 | 0.34657 | −1.43581 | +1.78238 | −0.05550 |
| 10 | 9.11449 | 7.25369 | +1.86080 | +0.02292 |
| 30 | 56.49559 | 54.71945 | +1.77614 | −0.06174 |
| 100 | 304.51714 | 302.79545 | +1.72169 | −0.11619 |
| 1000 | 5332.16420 | 5330.63329 | +1.53091 | −0.30696 |




**The prime fluctuation shadow vs. an off-line rogue wave.** The cyan curve plots exact computed values of $G_z(x) - \ln(2\pi)$, staying calmly near zero inside the dashed green $\pm 0.05\sqrt{x}$ envelope. By contrast, a hypothetical off-line zero at $\sigma = 0.78$ (magenta MODEL wave $-2\,\mathrm{Re}(x^\rho/(\rho(\rho-1)))$) rapidly bursts right through the $\sqrt{x}$ barrier.

The formalization first establishes that $G_z(x)$ is well-behaved and Lipschitz continuous—it can never spike arbitrarily steeply:

**Theorem 2.1** (Lipschitz continuity and linear bound for $G_z$) LEAN ✓

There exists a constant $K \geq 0$ such that for all real numbers $t, t' \geq \frac{1}{2}$,

$$|G_z(t)| \leq Kt \qquad \text{and} \qquad |G_z(t) - G_z(t')| \leq K|t - t'|.$$

LEAN › Formalism.zetaPrimeSumErrorLipschitz

**Proof.** Starting from $\sum_{n \leq k} \ln n = \sum_{n \leq k} \Lambda(n)\lfloor k/n \rfloor$ and Chebyshev's upper bound on $\psi(k)$, one obtains Mertens' bound $\sum_{n \leq k} \Lambda(n)/n \leq \ln k + \ln 4 + 4$. Writing $G_z(x)$ piecewise linearly minus $x \ln x$ bounds every local derivative by $\ln 4 + 7$, yielding a uniform Lipschitz constant on $[1/2, \infty)$. The full proof is checked by the Lean kernel. ∎

## 2.2 The Mellin Transform and Landau's Trap: Why a One-Sided Bound Kills Complex Zeros

Now we wire this real-line prime ledger back to $\zeta(s)$ in the complex plane. The connecting cable is the **Mellin transform**: integrate a function of $x \in (1,\infty)$ against $x^{-s-1}$. For any real sequence $a(n)$ bounded by $K\Lambda(n)$, the smoothed sum $S_a(x) = \sum_{1\le n\le x} a(n)(x/n-1)$ transforms into an exact Dirichlet series divided by $s(s-1)$:

**Theorem 2.2** (Mellin transform of generalized von Mangoldt sums) LEAN ✓

Suppose $a : \mathbb{N} \to \mathbb{R}$ and $K \in \mathbb{R}$ satisfy $|a(n)| \le K\Lambda(n)$ for all $n$. Then $S_a$ is measurable, $\int_1^\infty |S_a(x)|\, x^{-\sigma-1}\, dx < \infty$ for all $\sigma > 1$, and for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$,

$$\int_1^\infty S_a(x)\, x^{-s-1}\, dx = \frac{1}{s(s-1)} \sum_{n=1}^\infty \frac{a(n)}{n^s}.$$

LEAN › Formalism.mellinSgen

*Proof.* Interchange summation and integration by Fubini/dominated convergence. Each single term $\int_n^\infty (x/n-1)x^{-s-1}\,dx$ evaluates in closed form to $\frac{n^{-s}}{s(s-1)}$ for $\mathrm{Re}(s) > 1$, and summing over $n$ yields the $L$-series divided by $s(s-1)$. The full proof is checked by the Lean kernel. ∎

For $a(n) = \Lambda(n)$, the series is $-\zeta'(s)/\zeta(s)$. Subtract the Mellin transforms of $x \ln x$ and $-(1+\gamma)x$, and the pole at $s = 1$ cancels clean away! What remains is a meromorphic function whose **only** poles in $\mathrm{Re}(s) > 0$ sit at the zeros of $\zeta(s)$.

Now comes Edmund Landau's masterstroke—the **non-negative integral trap (Landau's Lemma)**.

Picture walking westward along the real axis from $b$ toward $a$, carrying a function $H(x) \ge 0$ whose values are **never negative**. Expand its Mellin integral $\Phi(s) = \int_1^\infty H(x)\, x^{-s-1}\, dx$ in a Taylor series around a real point $s_0 > b$ toward $s_0 - r$:

$$\Phi(s_0 - r) = \sum_{k=0}^\infty \frac{(-r)^k}{k!}\Phi^{(k)}(s_0) = \sum_{k=0}^\infty \frac{r^k}{k!}\int_1^\infty (\ln x)^k H(x)\, x^{-s_0-1}\, dx.$$

Look at the signs: $(\ln x)^k \ge 0$ and $H(x) \ge 0$ **leaves zero room for cancellation**! Every term in the Taylor series is positive. So as long as $\Phi(s)$ continues analytically along the real interval $(a, b]$, that all-positive series converges—which forces the integral itself to converge everywhere in the half-plane $\mathrm{Re}(s) > a$. And a convergent integral cannot blow up at a complex pole!

> When every entry in the ledger has the same sign, complex zeros lose their camouflage: unable to hide behind cancellation on the real axis, they cannot exist off the real axis either.

Lean proves this analytic backbone cleanly:

**Theorem 2.3** (Landau's non-negative integral lemma) LEAN ✓

Given the Mellin differentiation rule `MellinDeriv`: if $a < b$, $H : \mathbb{R} \to \mathbb{R}$ is measurable on $(1,\infty)$ with $H(x) \ge 0$, $\int_1^\infty H(x)x^{-\sigma-1}\,dx < \infty$ for all $\sigma > b$, and an analytic function $\Phi$

agrees with that integral for $\mathrm{Re}(s) > b$ while extending analytically to the real interval $(a, b]$, then for every $\sigma > a$,

$$\int_1^\infty H(x)\, x^{-\sigma-1}\, dx < \infty.$$

LEAN › Formalism.landauLemma_of

**Proof.** Suppose for contradiction that the abscissa of convergence $\sigma_c$ lies in $(a, b]$. Expanding $\Phi$ at a point slightly right of $\sigma_c$ turns $(-1)^k \Phi^{(k)}$ into the non-negative integral $\int_1^\infty (\ln x)^k H(x) x^{-s_0-1}\, dx$. Swapping summation and non-negative integration pushes integrability past $\sigma_c$ to some $\sigma' < \sigma_c$, contradicting the definition of $\sigma_c$. The full proof is checked by the Lean kernel. ∎

Here is why Landau's lemma is pure magic for $G_z(x)$: we do **not** even need a two-sided bound $|G_z(x)| \le Cx^\theta$. A **one-sided bound** alone—either $G_z(x) \ge -Cx^\theta$ or $G_z(x) \le Cx^\theta$—creates a non-negative function

$$H(x) = Cx^\theta + s_0\, G_z(x) \ge 0 \qquad (s_0 \in \{1, -1\}).$$

On the real interval $(0, 1)$, it is elementary that $\zeta(\sigma) \neq 0$ (since $(1 - 2^{1-\sigma})\zeta(\sigma) = 1 - 2^{-\sigma} + 3^{-\sigma} - \cdots > 0$ is a decreasing alternating series). So $\Phi$ is analytic along the whole real segment $(\theta, 1]$, Landau's trap snaps shut, and $\zeta(s)$ cannot vanish anywhere in $\mathrm{Re}(s) > \theta$!

**Theorem 2.4** (One-sided prime-sum bound implies half-plane nonvanishing) LEAN ✓

Let $s_0 \in \{1, -1\}$ and $0 < \theta < 1$. If there exist constants $C, x_0 \in \mathbb{R}$ such that for all $x \ge x_0$,

$$s_0\, G_z(x) \le C\, x^\theta,$$

then $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \theta$.

LEAN › Formalism.zetaLandau

**Proof.** Absorb the bounded values of $|G_z(x)|$ on $[1, \max(x_0, 1)]$ into $C$ to obtain a function $H(x) = C'x^\theta - s_0 G_z(x) \ge 0$ for all $x \ge 1$. Using `zetaRealNonvanish` ($\zeta(\sigma) \neq 0$ on $(0, 1)$) together with Theorem 2.3 and the Mellin representation Theorem 2.2, the integral extends analytically to $\mathrm{Re}(s) > \theta$, precluding any logarithmic pole from a zero of $\zeta(s)$ in that half-plane. The full proof is checked by the Lean kernel. ∎

**Corollary 2.5** (Epsilon-tolerant Landau criterion) LEAN ✓

Let $s_0 \in \{1, -1\}$ and $0 \le \theta < 1$. If for every $\varepsilon > 0$ there exist $C, x_0$ such that $s_0\, G_z(x) \le C\, x^{\theta+\varepsilon}$ for all $x \ge x_0$, then $\zeta(s) \neq 0$ for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \theta$.

LEAN › Formalism.zetaLandauEps

**Proof.** For any $s$ with $\mathrm{Re}(s) > \theta$, pick $\varepsilon = \min(\mathrm{Re}(s) - \theta, 1 - \theta)/2 > 0$ and apply Theorem 2.4 at threshold $\theta + \varepsilon < \mathrm{Re}(s)$. The full proof is checked by the Lean kernel. ∎

And the same mechanism generalizes cleanly to Dirichlet character sums and complex linear combinations (`OCStep`, `FamilyLandauC` in `StatementOC.lean` / `StatementFL.lean`): **whoever pins down the one-sided fluctuation to $x^{1/2+\varepsilon}$ clears the entire right half-plane.**

VIZ

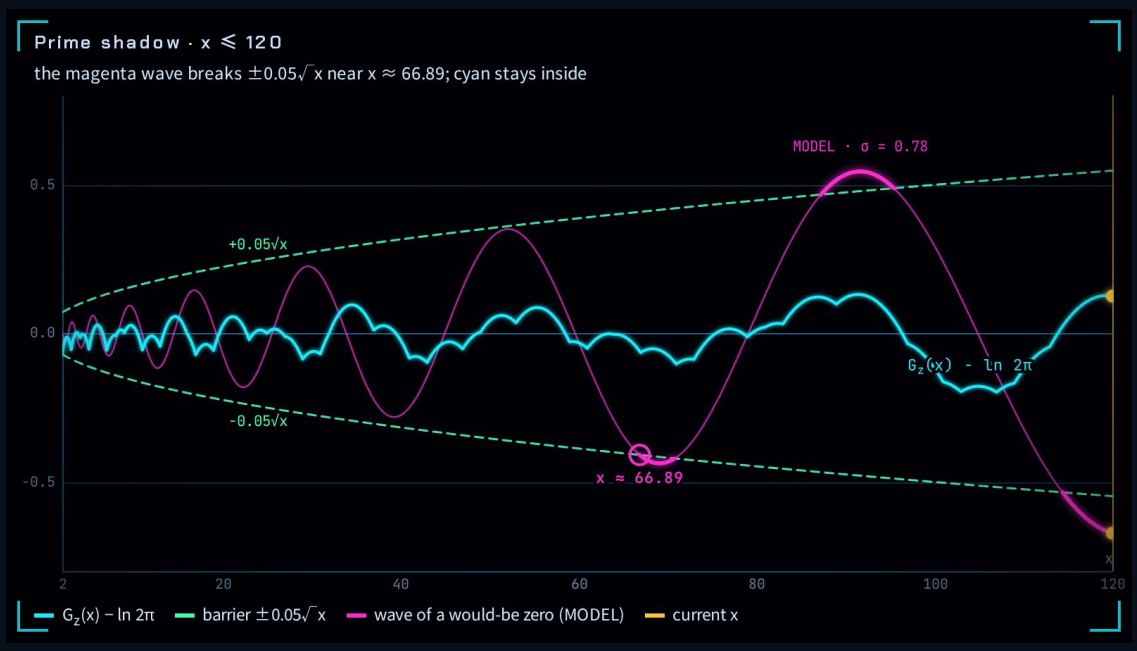



**Figure 3** ZETA//MIRROR · Critical mirror · Re(s) = 1/2 mirrors every off-line zero into a twin, and an off-line zero would break the √x barrier

**Landau explicit-formula rogue wave:** The cyan curve computes $G_z(x) - \ln(2\pi)$ directly from $\Lambda(n)$ ($G_z(10) \approx 1.86080$, $G_z(100) \approx 1.72169$), remaining inside the dashed $\pm 0.05\sqrt{x}$ envelope, while the magenta MODEL wave for $\sigma = 0.78, t = 11.13$ breaks through near $x \approx 66.89$ and swings to $0.672 > 0.548$ by $x = 120$.
Interactive version:
http://rh.trureturing.org/riemann-hypothesis/viz/critical-mirror/

## 2.3 Where Classical Weapons Stall: de la Vallée Poussin and Siegel

Before we unveil the genuinely new engines in Chapters 3–6, let us be honest about two classical pillars built more than a century ago. Neither one comes anywhere near $\mathrm{Re}(s) = \frac{1}{2}$ on its own, yet both are formalized in full inside the repository because the later machinery needs them as local scaffolding.

The first is the 1896 Hadamard–de la Vallée Poussin zero-free region, built on the humble trigonometric inequality

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geq 0.$$

It pushes zeros away from $\mathrm{Re}(s) = 1$, yet as $|\mathrm{Im}(s)|$ grows the protected strip hugs the wall $\mathrm{Re}(s) = 1$ ever tighter and never approaches $\frac{1}{2}$:

**Theorem 2.6 (Classical zero-free region for $\zeta(s)$)** LEAN ✓
There exists a constant $c > 0$ such that for all $s \in \mathbb{C}$ with $s \neq 1$ and

$$\mathrm{Re}(s) \geq 1 - \frac{c}{\ln(|\,\mathrm{Im}(s)| + 2)},$$

we have $\zeta(s) \neq 0$.

LEAN › Formalism.ZF.zeroFreeRegion

**Proof.** Combines the local logarithmic-derivative expansion `landauLocal` with Mertens' prime estimate, the $3 + 4\cos\theta + \cos 2\theta \geq 0$ positivity inequality, growth bounds in the right half-plane, and an explicit check at bounded heights. The full proof is checked by the Lean kernel. ∎

The second classical pillar is Carl Ludwig Siegel's 1935 lower bound on $L(\chi, 1)$ for real quadratic characters—the shield against possible Landau–Siegel real zeros close to $s = 1$, without which averaging over arithmetic progressions would lose control of bad moduli $q$:

**Theorem 2.7** (Siegel's lower bound for $L(1, \chi)$) LEAN ✓

For every $\varepsilon > 0$, there exists a constant $c > 0$ such that for every positive modulus $q \geq 1$ and every nontrivial real Dirichlet character $\chi$ modulo $q$ ($\chi^2 = 1$, $\chi \neq 1$),

$$\|L(\chi, 1)\| \geq c\, q^{-\varepsilon}.$$

LEAN › Formalism.ZeroFreeDirichlet.siegelLower

**Proof.** Constructs an auxiliary product $L$-function whose coefficients satisfy the key inequality `bqKey`, then splits by whether an exceptional real zero close to $1$ exists. As classically, the constant $c$ is ineffective, which suffices for the Siegel–Walfisz-type prime estimates needed later. The full proof is checked by the Lean kernel. ∎

So the classical toolbox is on the table. Why did trigonometric positivity and single log-derivative bounds get stuck just left of $\sigma = 1$ for over a century?

Because $3 + 4\cos\theta + \cos 2\theta \geq 0$ only uses **first-order positivity**—it sees only single ripples of primes and has no way to enforce **high multiplicity** over the integers.

> Old methods hit a wall not because primes are too wild, but because they try to measure an ocean with a one-inch ruler.

In the next chapter, we look at the finite-field miracle that breaks that wall—Stepanov's high-multiplicity polynomial trick—and see how the proof forces the stubborn integers to submit to differentiation.

VIZ

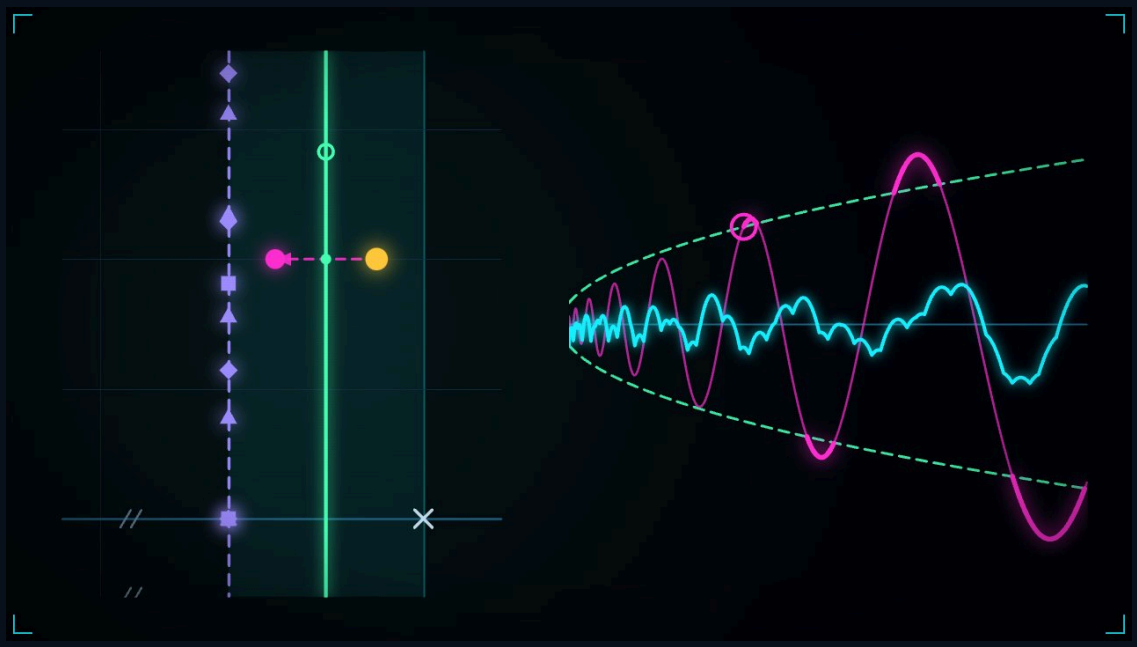


**Figure 4** ZETA//MIRROR · Critical mirror · Re(s) = 1/2 mirrors every off-line zero into a twin, and an off-line zero would break the √x barrier

**The full mirror and prime-shadow lab:** Switch freely between the complex $s$-plane mirror and the computed $G_z(x) - \ln(2\pi)$ fluctuation view, drag the test zero $\rho = \sigma + it$ across the critical strip, and explore all four presets (`on-line`, `off-line`, `imprimitive`, `landau-trap`).
Interactive version:
http://rh.trureturing.org/riemann-hypothesis/viz/critical-mirror/

§ 03

# Chapter 3 Differentiating the Integers: Stepanov's Determinant and the Dream of Multiplicity

> Give a mathematician polynomials, and derivatives come free with roots to count; hand them whole numbers, and the stone refuses to bend—until you embed the integers inside geometry, where even stones grow tangent lines.

Why was the Riemann Hypothesis over finite-field curves solved decades ago while the integers held out for 160 years?

Replace $\mathbb{Z}$ with a curve over a finite field $\mathbb{F}_q$ (such as $y^2 = x^3 + x$). There, the analogue of RH fell first to André Weil, later to Pierre Deligne, and—in 1969—to a shockingly short, elementary proof by Sergei Stepanov that needed barely a handful of pages of high-school-level algebra.

Stepanov's whole magic trick comes down to a single word: **multiplicity**.

## 3.1 Stepanov's Trick: Why High-Order Zeros Beat Point Counting

Here is the core idea in thirty seconds. Suppose you want to bound how many rational points lie on a curve. If you build an auxiliary polynomial $P(x)$ that simply equals $0$ at each point (multiplicity $m = 1$), each point uses up $1$ unit of your degree budget $\deg P$, leaving you with only the trivial bound.

Now suppose you force $P(x)$ at every rational point to vanish not only at $P(x_0) = 0$, **but also at $P'(x_0) = P''(x_0) = \ldots = P^{(m-1)}(x_0) = 0$!**

Suddenly every single point counts $m$ times against the degree of $P$. As long as $P$ is not identically zero, one line closes the trap:

$$m \times (\text{number of points}) \leq \deg P.$$

Make $m$ large, and the upper bound drops by a factor of $1/m$. Imagine a banquet table 100 inches long: if every guest takes 1 inch, you can seat 100 guests; if you require every guest to occupy 10 inches side by side, at most 10 guests can fit in the room.

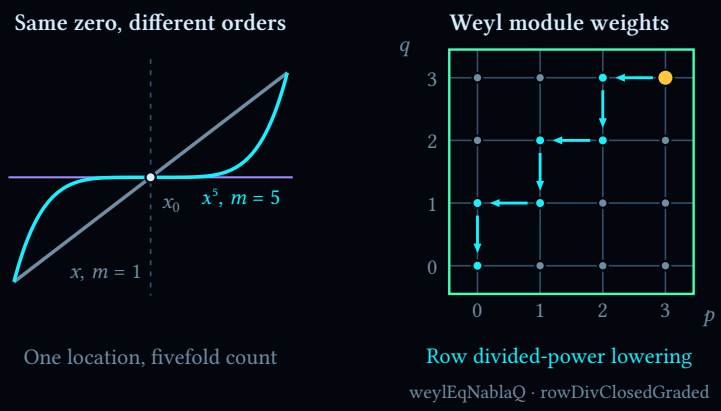
> Simple zeros police the doorway; only high multiplicity clears the room. Yet outside polynomial rings, who lets you differentiate a whole number?

Here sits the **discrete wall** that blocked analytic number theory for a century and a half. In polynomials over a field, differentiation $d/dx$ is free. Over the integers $\mathbb{Z}$, $n$ and $n+1$ sit one stubborn unit apart: you cannot let $h \to 0$, so ordinary calculus falls through the floor.

## 3.2 Manufacturing Derivatives Out of Integers: Quadrics, CM Curves, and Graded Weyl Divisibility

To force whole numbers to yield multiplicity $m$, the proof engineers two complementary mechanisms that secretly speak the same geometric language:

1. **The geometric disguise: path quadrics and CM curves.** Sums over Dirichlet characters are converted, via repeated Parseval identities and "path quadrics" $x_1x_2 + x_2x_3 + \cdots + x_{k-1}x_k$, into bilinear pairings; and on elliptic curves with complex multiplication (CM), the invariant derivation $D$ satisfies $D^p \equiv 0 \pmod{p}$ at inert primes, acting as a built-in algebraic differentiation switch that forces divisibility by prime powers.
2. **The algebraic backbone: divided powers and Fock positivity.** Whenever you build an object with "multiplicity $m$" over integers rather than complex numbers, ordinary derivatives $\frac{d^k}{dx^k}$ spit out nasty factorials $k!$ that tangle denominators. You have to replace derivatives with **divided-power operators** (`rowDiv`) $\frac{1}{k!}D^k$ and prove that the integral Weyl module stays locked tightly to its symmetric dual equations!




**Left: simple crossing vs. multiplicity-5 tangency; Right: graded Weyl weights and the Fock staircase.** On the left, the cyan quintic $(x - x_0)^5$ hugs the axis flat so one root consumes five units of degree. On the right, starting from the amber highest-weight corner, divided-power row operators `rowDiv` step cleanly down the grid while staying inside the green Weyl boundary under the cyan Fock inner product.

Let us look right inside this algebraic gearbox (`GradedWeylDivisibility`) with bare hands. Picture an $r \times n$ matrix of variables $X = (x_{a,j})$ where columns carry weight $\nu$ and rows carry symmetry. How do you guarantee that the span generated from the highest-weight vector by divided-power row operators (`weyl`) matches the kernel of column-symmetry equations (`dualWeyl`)—not only over $\mathbb{Q}$, but up to controlled integer multipliers over $\mathbb{Z}$?

Enter an old friend from quantum mechanics: the **Fock (Fischer) inner product**! Given two rational polynomials $f, g \in \mathbb{Q}[X]$, define

$$\langle f, g \rangle_{\mathrm{Fock}} = \sum_m f_m \, g_m \prod_k (m_k!),$$

summing over monomials $m$ weighted by the product of factorials of their exponents! Just like creation and annihilation operators on a harmonic oscillator, multiplication by

$x_k$ and differentiation $\partial/\partial x_k$—and more generally opposite divided-power row operators `rowDiv a b k` and `rowDiv b a k`—are **exact adjoints** of each other under this positive-definite pairing!

Five crisp theorems verified in `FinalB.lean` lock this gearbox shut:

**Lemma 3.1** (Symmetry and strict positivity of the rational Fock pairing) LEAN ✓

For any index type $\sigma$ and any multivariate polynomials $f, g \in \mathbb{Q}[X_\sigma]$, the rational Fock pairing is symmetric, $\langle f, g\rangle_{\mathrm{Fock}} = \langle g, f\rangle_{\mathrm{Fock}}$, and whenever $f \neq 0$,

$$\langle f, f\rangle_{\mathrm{Fock}} > 0.$$

LEAN › Formalism.GradedWeylDivisibility.fockQPos

**Proof.** In $\sum_m f_m^2 \prod_k m_k!$, every factorial product is strictly positive over $\mathbb{Q}$; if $f \neq 0$, at least one monomial coefficient $f_m$ is nonzero, making the sum strictly positive. The full proof is checked by the Lean kernel. ∎

**Lemma 3.2** (Fock adjointness of divided-power row operators) LEAN ✓

For all $r, n \in \mathbb{N}$, all row indices $a, b \in \mathrm{Fin}\, r$, all $k \in \mathbb{N}$, and all $f, g \in \mathbb{Q}[X_{r\times n}]$,

$$\langle \mathrm{rowDiv}^{(k)}_{a,b}(f),\, g\rangle_{\mathrm{Fock}} = \langle f,\ \mathrm{rowDiv}^{(k)}_{b,a}(g)\rangle_{\mathrm{Fock}}.$$

LEAN › Formalism.GradedWeylDivisibility.fockQRowDivAdj

**Proof.** Verified directly on monomial bases by matching the binomial factors produced when transferring $k$ powers from row $b$ to row $a$ against those from row $a$ to row $b$ weighted by exponent factorials. The full proof is checked by the Lean kernel. ∎

**Lemma 3.3** (Divided-power closed subspaces are row-weight graded) LEAN ✓

Let $M \subseteq \mathbb{Q}[X_{r\times n}]$ be a $\mathbb{Q}$-submodule closed under all divided-power row operators $\mathrm{rowDiv}^{(k)}_{a,b}$. Then for every $f \in M$ and every row weight $\mu : \mathrm{Fin}\, r \to \mathbb{N}$, the $\mu$-homogeneous component of $f$ also belongs to $M$.

LEAN › Formalism.GradedWeylDivisibility.rowDivClosedGraded

**Proof.** Scaling individual rows over the infinite field $\mathbb{Q}$ separates row-weight components via Vandermonde interpolation, and invariance under diagonal scaling is preserved by divided-power closure. The full proof is checked by the Lean kernel. ∎

Put those three pieces together and any mismatch between `weyl` and `dualWeyl` vanishes on the spot: if `weyl` were strictly smaller than `dualWeyl`, its Fock-orthogonal complement would be non-empty; by positivity and adjointness, climbing by raising operators inside that complement would produce a second independent highest-weight line—contradicting the uniqueness of the highest-weight line!

**Theorem 3.4** (Coincidence of rational Weyl and dual Weyl modules) LEAN ✓

For all natural numbers $n \le r$ and every antitone weight $\nu : \mathrm{Fin}\, n \to \mathbb{N}$, the rational Weyl and dual Weyl modules coincide:

$$\mathrm{weyl}_{\mathbb{Q}}(r, n, \nu) = \mathrm{dualWeyl}_{\mathbb{Q}}(r, n, \nu).$$

LEAN › Formalism.GradedWeylDivisibility.weylEqNablaQ

**Proof.** Under the positive-definite Fock pairing (Lemma 3.1), `weyl` is the smallest divided-power closed submodule containing the highest-weight vector. By adjointness (Lemma 3.2) and grading (Lemma 3.3), any nonzero element in its orthogonal complement inside `dualWeyl` raises to a nonzero highest-weight vector orthogonal to `hw`, contradicting highest-weight uniqueness (`hwLine`). The full proof is checked by the Lean kernel. ∎

**Corollary 3.5** (Integral clearing of denominators for dual Weyl elements) LEAN ✓

For all $n \le r$, every antitone weight $\nu$, and every integral element $f \in \mathrm{dualWeyl}_{\mathbb{Z}}(r, n, \nu)$, there exists a nonzero integer $c \neq 0$ such that

$$c \cdot f \in \mathrm{weyl}_{\mathbb{Z}}(r, n, \nu).$$

LEAN › Formalism.GradedWeylDivisibility.weylRational

**Proof.** Embed $f$ into the rational module $\mathrm{dualWeyl}_{\mathbb{Q}}$, express it in the rational Weyl basis via Theorem 3.4, and clear a common integer denominator $c \neq 0$ back into $\mathrm{weyl}_{\mathbb{Z}}$. The full proof is checked by the Lean kernel. ∎

See what this machine guarantees? **No matter how high you stack algebraic multiplicity over $\mathbb{Z}$, your integral structures never spring a rational-denominator leak.**

Yet having a leak-proof algebraic container still leaves the central creative mystery unanswered: when staring at a von Mangoldt prime sum of length $X$, where do we find a machine that actually manufactures $m \sim \sqrt{X}$ layers of multiplicity natively in the automorphic world?

> When a single space cannot bear the multiplicity you need, do not force one ruler harder—chain a sequence of mirrors so each reflection buys one more order of vanishing.

In the next chapter, we meet the signature leap of the entire proof: chaining $\sqrt{X}$ unitary theta lifts end to end.

§ 04

# Chapter 4 A Chain of $\sqrt{X}$ Mirrors: Iterated Theta Lifts and the Seesaw Law

> One mirror reflects you once; line up $\sqrt{X}$ mirrors in a corridor, and every bounce hands you one more order of differentiation—then a quick flip of the seesaw at the far end unties the whole bundle of characters at once.

Arguably the single most electric line in the author's announcement note asks one simple design question: given a von Mangoldt prime sum of length $X$, how many theta lifts should you chain together so that each link simulates one more derivative inside the automorphic world?

The answer:

$$m \sim \sqrt{X}.$$

Step back for a second and ask *why* $\sqrt{X}$ is the magic scale.

Inside a prime sum of length $X$, $X$ integers are pushing against your bound. A tiny fixed multiplicity like $m = 2$, $m = 3$, or even $\ln X$ has nowhere near enough leverage to pin the ripples down to $\sigma = \frac{1}{2}$—it is like trying to hold back a tide with a coffee spoon. Go too far the other way, say $m \sim X$, and the cumulative height, level, and covolume cost of the lifts blows up your budget long before you finish! Only right along the knife-edge $m \sim \sqrt{X}$ do the gains of multiplicity and the cost of lifting balance into a straight shot toward $\frac{1}{2}$.

## 4.1 What Is a Theta-Lift Chain? Each Reflection Buys One More Derivative

First let us strip the scary phrase "iterated theta lifts between totally definite unitary groups" down to plain sight.

Paradoxically, "totally definite" is the **friendliest, tightest** kind of unitary group you could ask for. Unlike noncompact hyperbolic groups that shoot off to infinite cusps, a totally definite unitary group is compact at every infinite place—like a smooth closed sphere. Automorphic forms on it are literally just functions evaluated on finitely many equivalence classes of Hermitian lattices!

What does a single theta lift $U(X_j) \to U(X_{j+1})$ actually do? Think of it as a two-sided projection machine: on the left sits a function $f$ living on the lattices of $U(X_j)$; in the

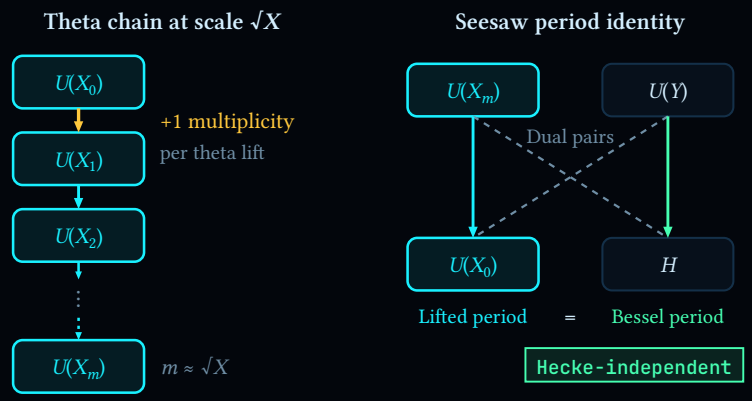
middle sits a theta kernel matrix $K_j(i, y)$; summing across the finite set of lattice class representatives sends $f$ to

$$(\text{lifted function})(y) = \sum_i w_i \, f(i) \, K_j(i, y).$$

In representation-theoretic terms, each step replaces $\pi$ by $\pi \boxplus \mu_j[t]$, appending one more layer of multiplicity tied to a Hecke character $\mu_j$. Chain $m$ of these kernels end to end:

$$U(X_0) \longrightarrow U(X_1) \longrightarrow U(X_2) \longrightarrow \cdots \longrightarrow U(X_m),$$

and you manufacture **automorphic multiplicity of order $m$** without ever begging the integers for a calculus derivative they cannot give!




**Left: the $\sqrt{X}$-step iterated theta-lift chain; Right: the Seesaw diamond converting lifted periods to Hecke-independent Bessel periods.** On the left, each green theta kernel adds one step of multiplicity along the cyan chain of definite unitary groups (amber marks the active transfer link) until reaching $U(X_m)$. On the right, the Seesaw identity flips the left-leg lifted period—which still drags along the Hecke characters—into an equal right-leg Bessel period free of those characters.

Inside `Formalism/ThetaLiftIndependence`, `ChainScalar.lean`, `LatLiftIndep.lean`, and `FinalThetaLiftAMFKIndep.lean` weld three critical invariants of this chain in crisp finite-dimensional linear algebra:

1. **The chain stays inside integral/lattice submodules** all the way to step $m$;
2. **The lift sum is completely independent of which lattice class representatives you pick**;
3. **Finite-field Gauss operators invert cleanly** up to the field cardinality $|\mathbb{F}|$.

**Lemma 4.1** (Adjoint stability along an $m$-step iterated chain) LEAN ✓

Let $T_j : E_{j+1} \to E_j$ be a family of linear maps between finite-dimensional inner product spaces, and let $S_j \leq E_j$ be submodules. If every single step satisfies $T_j^*(x) \in S_{j+1}$ for all $x \in S_j$, then for every chain length $m \in \mathbb{N}$ and every $x \in S_0$,

$$(\text{downChain}_T(m))^*(x) \in S_m.$$

LEAN › `Formalism.ThetaLiftIndependence.ChainScalarLemmas.mem`

*Proof.* By induction on $m$: at $m = 0$ the adjoint of the identity keeps $x \in S_0$; for $m + 1$, the rule $(A \circ B)^* = B^* \circ A^*$ splits the step into the length-$m$ adjoint followed by $T_m^*$, preserving membership in $S_{m+1}$. The full proof is checked by the Lean kernel. ∎

**Theorem 4.2** (Lattice-lift sums do not depend on class representatives) LEAN ✓

Let $K$ be a number field, $V$ a $K$-vector space, and $h$ a Hermitian form on $V$. Whenever $f$ and $\Theta$ are invariant under the isometry group $\mathrm{Isom}(h)$, any two valid sets of class representatives $s, s'$ for a lattice $L$ yield the exact same lift sum:

$$\mathrm{latLift}(h, s, f, \Theta) = \mathrm{latLift}(h, s', f, \Theta).$$

LEAN › Formalism.ThetaLiftIndependence.latLiftIndep

*Proof.* Any two systems of class representatives $s$ and $s'$ stand in explicit bijection via elements of $\mathrm{Isom}(h)$ (uniqueness of paired representatives uses invariance of stabilizers under conjugation); since weights, $f$, and $\Theta$ are isometry-invariant, both sums match term by term. The full proof is checked by the Lean kernel. ∎

**Theorem 4.3** (Duality and inversion of finite-field Gauss operators) LEAN ✓

Let $\mathbb{F}$ be a finite field, $\psi$ a nontrivial additive character of $\mathbb{F}$, and $\rho : \mathbb{F}^\times \to \mathrm{End}_{\mathbb{C}}(V)$ a representation. For every vector $v \in V$ satisfying $\sum_{c\in\mathbb{F}^\times} \rho(c)v = 0$, the Gauss operator and its dual recover $v$ scaled by $|\mathbb{F}|$:

$$\mathrm{gaussOp}(\psi,\rho)\big(\mathrm{gaussOpDual}(\psi,\rho)(v)\big) = |\mathbb{F}|\cdot v, \qquad \mathrm{gaussOpDual}(\psi,\rho)\big(\mathrm{gaussOp}(\psi,\rho)(v)\big) = |\mathbb{F}|\cdot v.$$

LEAN › Formalism.ThetaLiftIndependence.final_gaussOp

*Proof.* Expanding the double sum and applying additive character orthogonality produces $|\mathbb{F}| - 1$ on the diagonal $a = 1$ plus $-1$ times $\sum_{c\neq 0, c\neq 1} \rho(c)v$; using $\sum_{c\in\mathbb{F}^\times} \rho(c)v = 0$ turns that sum into $+1 \cdot v$, yielding $|\mathbb{F}| \cdot v$. The full proof is checked by the Lean kernel. ∎

## 4.2 The Seesaw Law, Big-Cell Minors, and Formal Degrees

Suppose you have built all $m \sim \sqrt{X}$ links of the chain up to $U(X_m)$. An immediate headache remains: **the period integral on $U(X_m)$ is still tangled up with the sequence of auxiliary Hecke characters $\mu_1, \ldots, \mu_m$!** As $m$ grows, tracking character twists inside the integral looks fatal.

How do you snap off those clinging Hecke characters in one motion?

With the **Seesaw law** (Rallis inner product identity).

Think of a literal playground seesaw: if a heavy kid sits on the left end and a lighter kid balances on the right end, you do not have to wedge a bathroom scale under the heavy kid—you can measure the counterweight sitting on the opposite leg! In a diamond of dual reductive pairs:

- the **left leg** measures the period of the $m$-fold iterated lift, which explicitly carries the Hecke characters;
- the **right leg** swaps the pairing across the diamond and reads the exact same number as a **Bessel period** on a companion group—which turns out to be **completely independent of the Hecke characters**!

To make that seesaw balance quantitatively—so lower bounds on norm and divisibility propagate cleanly back down the chain—two exact structural calculations do the heavy

lifting: the principal minors on the big cell (`bigCellMinors`) and the Harish-Chandra formal degree (`fdegDimV`), both verified in `Formalism/FormalDegreeBessel`:

**Theorem 4.4 (Leading principal minors on the big cell)** LEAN ✓

For $k, r \in \mathbb{N}$, vectors $z, z' \in \mathbb{C}^{2r+k+1}$ satisfying the isotropic frame hypothesis `inWb`, diagonal parameters $c \in \mathbb{C}^r$, and unitary block $\mu \in M_{k+1}(\mathbb{C})$, the big-cell matrix `bigCell k r c μ` satisfies

$$D_l\big(\mathrm{bigCell}(k, r, c, \mu)\big) = \prod_{a<l} c_a \qquad (0 \le l \le r),$$

with ratio minors recovering $c_a$ when every $c_a \neq 0$, and the pairing zeta function reducing to the canonical Hermitian form.

LEAN › `Formalism.QuantumDeterminants.bigCellMinors`

**Proof.** Under the `inWb` isotropic basis splitting, `bigCell` factors cleanly into upper/lower block triangular components and the central unitary block $\mu$, making every $l$-th leading principal determinant an immediate product of the first $l$ entries $c_a$. The full proof is checked by the Lean kernel. ∎

**Theorem 4.5 (Harish-Chandra formal degree formula)** LEAN ✓

For every $n \in \mathbb{N}$ and every strictly decreasing Harish-Chandra parameter $\lambda_0 > \lambda_1 > \ldots > \lambda_{n-1}$, the formal degree equals the Vandermonde product divided by consecutive factorials:

$$\mathrm{fdeg}(\lambda) = \frac{\prod_{0 \le a < b < n} (\lambda_a - \lambda_b)}{\prod_{i=0}^{n-1} i!}.$$

LEAN › `Formalism.QuantumDeterminants.fdegDimV`

**Proof.** Strict monotonicity turns every $|\lambda_a - \lambda_b|$ into $\lambda_a - \lambda_b$, and reordering the product $\prod_{0 \le a < b < n} (b - a)$ yields $\prod_{i=0}^{n-1} i!$ (`formal_degree_factorial_holds`). The full proof is checked by the Lean kernel. ∎

Now we have a $\sqrt{X}$-step mirror chain and a seesaw that sheds the Hecke characters at the top. Surely the road to $\sigma = \frac{1}{2}$ is open?

Not yet. When you actually tally up the covolume and height cost of every lift step along a single character, you hit a brutal surprise—**the single-character budget stalls dead at $\sigma = \frac{2}{3}$!**

> Every single-handed strategy eventually hits an honest fractional ceiling. Two-thirds is not bad luck; it is the exact price limit of standing alone.

In the next chapter, we use lattice slopes and Bost's inequality to put that $2/3$ ceiling under a microscope—and then smash right through it by gluing integral structures across an entire family of Hecke characters.

VIZ

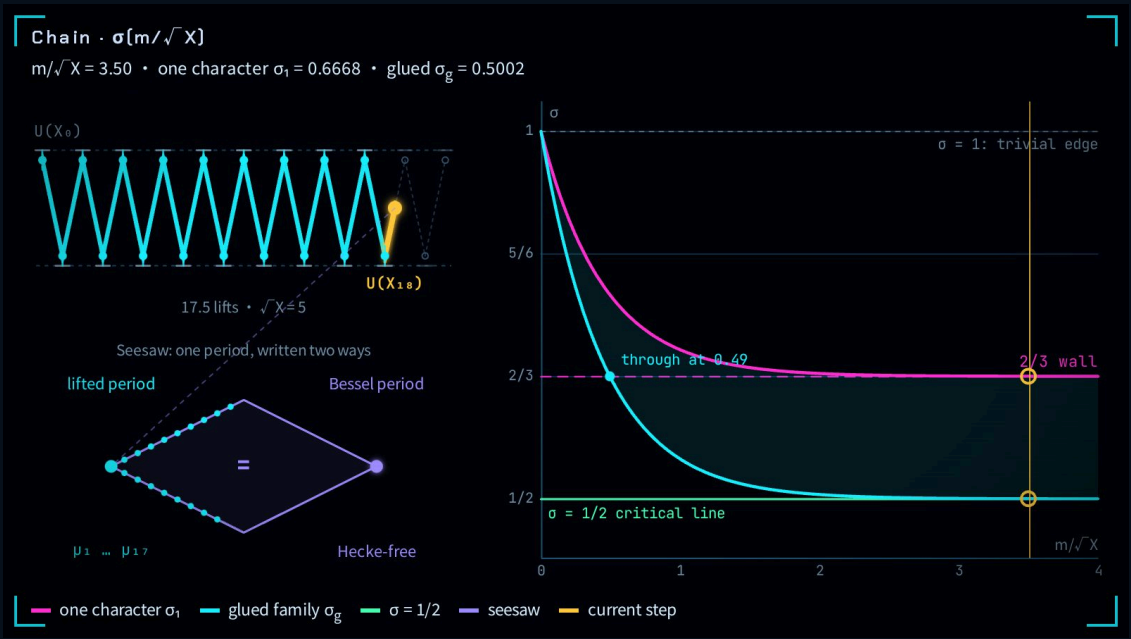



**Figure 5** THETA//SLOPE · Theta chain budget · a chain of √X mirrors stalls at 2/3 until a glued family flattens the roof

**The $\sqrt{X}$ mirror chain vs. the $2/3$ ceiling:** The zigzag mirror corridor $U(X_0) \to \ldots \to U(X_m)$ and violet seesaw rhombus trade a character-tangled lifted period for a clean Bessel period, while the MODEL threshold plot shows the glued-family curve crossing below $2/3$ down to $\sigma = 0.5002$ at $b = m/\sqrt{X} = 3.5$ as single-character lifting stalls at $2/3$. Interactive version:
http://rh.trureturing.org/riemann-hypothesis/viz/theta-chain-budget/

§ 05

# Chapter 5 Breaking the $2/3$ Ceiling: Lattice Slopes, Bost's Inequality, and Integral Gluing

> Pay freight separately for each character, and inflation eats your margin at two-thirds; glue an integral structure across the whole family along nearly orthogonal axes, and the last one-sixth cliff collapses down to one-half.

At the end of Chapter 4 we hit an honest mathematical wall: **push a single-character iterated theta-lift chain as hard as you want, and the budget stalls at $\sigma = \frac{2}{3}$.**

If you follow analytic number theory, the number $2/3$ probably rings a familiar bell—time and again across prime gaps, zero-density theorems, and convex bounds, single-variable arguments stall around $2/3$ or $3/4$. Here the reason is pure bookkeeping. Each theta lift buys you one extra layer of divisibility from formal degrees and lattice counts, **but simultaneously raises the level, norm, and covolume cost of the space**. Run the balance sheet for a single Hecke character, and gains minus costs top out right at $\sigma = \frac{2}{3}$.

To crack past $\frac{2}{3}$ down to $\frac{1}{2}$, the proof makes two decisive moves: **first measure every lift's exact cost with lattice slopes, then glue an integral structure across a whole family of Hecke characters so their costs share the load along nearly orthogonal directions.**

## 5.1 The Roof Over a Lattice: Harder–Narasimhan Slopes

Do not let terms like "Harder–Narasimhan filtration" and "Bost's slope inequality" push you off balance. Underneath, they are a piece of elementary geometry about roofs and short vectors.

Picture an integer lattice $L$ inside a Euclidean space $E$ as a warehouse full of grid points:

- its dimension is the **rank** $\mathrm{rk}(L)$;
- the volume of one fundamental box is its **covolume** $\mathrm{covol}(L)$: packed lattices have tiny covolumes, sparse lattices have huge ones.

To turn multiplication of volumes into addition, define the **degree** and **slope** of any nonzero sublattice $N \leq L$ by

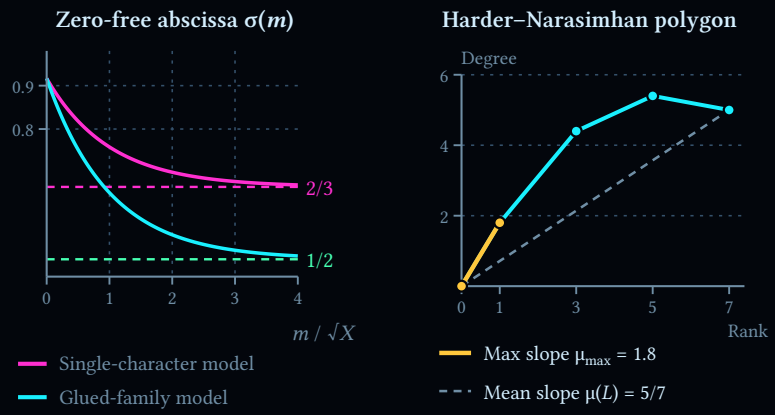
$$\deg(N) = -\ln \mathrm{covol}(N), \qquad \mu(N) = \frac{\deg(N)}{\mathrm{rk}(N)}.$$

Hold onto one crystal-clear picture: **a sublattice with densely packed points has tiny covolume, large degree, and a steep positive slope $\mu(N)$.** The supremum of slopes over all nonzero sublattices $N \leq L$ is the maximal slope $\mu_{\max}(L)$.

Try a four-step example with real numbers! Take a rank-7 lattice $L$ containing a nested chain of saturated sublattices $0 = L_0 \subset L_1 \subset L_2 \subset L_3 \subset L_4 = L$ at $(\mathrm{rk}, \deg)$ coordinates:

- $(0, 0) \to (1, 1.8)$: segment slope $\mu_1 = 1.8/1 = 1.8$ (one tight short vector!);
- $(1, 1.8) \to (3, 4.4)$: segment slope $(4.4 - 1.8)/(3 - 1) = 1.3$;
- $(3, 4.4) \to (5, 5.4)$: segment slope $(5.4 - 4.4)/(5 - 3) = 0.5$;
- $(5, 5.4) \to (7, 5.0)$: segment slope $(5.0 - 5.4)/(7 - 5) = -0.2$.

Plot those five points in order and look at the polygon: the slopes $1.8 > 1.3 > 0.5 > -0.2$ step strictly downward, **forming an arched concave roof!** Even though the whole lattice has average slope $5.0/7 \approx 0.714$, its steepest first rafter pushes $\mu_{\max}(L) = 1.8$—and that steep slope forces a short nonzero lattice vector to exist!




**Left: single-character stalling at $2/3$ vs. glued-family descent to $1/2$ (MODEL); Right: exact Harder–Narasimhan concave slope polygon.** On the left, the magenta curve shows naive single-character lifting flattening against the $\sigma = 2/3$ ceiling, while the cyan glued curve breaks through toward the green $\sigma = 1/2$ line. On the right, exact computed points $(0, 0), (1, 1.8), (3, 4.4), (5, 5.4), (7, 5.0)$ trace the concave roof, with the steepest first segment $\mu_{\max} = 1.8$ highlighted in amber against the dashed average slope $5.0/7 \approx 0.714$.

All four pillars of this lattice-slope geometry are proved from scratch inside `ExistsSaturatedSlopeEqMuMax.lean`:

**Theorem 5.1** (Boundedness of slopes and upper bound by $\mu_{\max}$) LEAN ✓

For any discrete submodule (lattice) $L \leq E$ in a finite-dimensional real inner product space $E$, the set of slopes $\mathrm{slopes}(L)$ is bounded above, and every nonzero sublattice $N \leq L$ satisfies

$$\mu(N) \leq \mu_{\max}(L).$$

LEAN › `Formalism.F.slopesBdd` `Formalism.F.slopeLeMuMax`

**Proof.** Using the Gram-determinant projection bound `gramDetLe`, every nonzero sublattice $N \leq L$ has covolume bounded uniformly away from zero by the minimal norm in $L$, bounding $-\ln \mathrm{covol}(N)/\mathrm{rk}(N)$ from above; the supremum inequality follows directly. The full proof is checked by the Lean kernel. ∎

**Theorem 5.2** (Minkowski short-vector theorem) LEAN ✓

For every nonzero discrete lattice $L \leq E$ in a finite-dimensional real inner product space, there exists a nonzero vector $v \in L$ satisfying

$$\|v\| \leq \sqrt{\mathrm{rk}(L)}\ \mathrm{covol}(L)^{1/\mathrm{rk}(L)}.$$

LEAN › `Formalism.F.shortVector`

**Proof.** Identifies `covol(L)` with Mathlib's measure-theoretic `ZLattice.covolume` (`covolEqCovolume`) and applies Minkowski's convex-body theorem on a cube scaled to radius $\mathrm{covol}(L)^{1/\mathrm{rk}(L)}$, converting $\ell^\infty$ norm to Euclidean norm via the factor $\sqrt{\mathrm{rk}(L)}$. The full proof is checked by the Lean kernel. ∎

**Theorem 5.3** (Additivity of rank and degree for saturated sublattices) LEAN ✓

Let $L \leq E$ be a discrete lattice and $N \leq L$ a saturated sublattice ($N = L \cap \mathrm{span}_{\mathbb{R}}(N)$), with orthogonal quotient lattice $\mathrm{quot}(L, N)$ projected onto $N^\perp$. Then $\mathrm{quot}(L, N)$ is discrete, and

$$\mathrm{rk}(L) = \mathrm{rk}(N) + \mathrm{rk}\big(\mathrm{quot}(L, N)\big), \qquad \deg(L) = \deg(N) + \deg\big(\mathrm{quot}(L, N)\big).$$

LEAN › `Formalism.F.deg_eq_deg_add_deg_quot`

**Proof.** Extending a $\mathbb{Z}$-basis of the saturated sublattice $N$ to a $\mathbb{Z}$-basis of $L$ puts the Gram matrix in block-triangular orthogonal form, giving $\mathrm{covol}(L) = \mathrm{covol}(N) \cdot \mathrm{covol}(\mathrm{quot}(L, N))$; taking negative logarithms yields degree additivity. The full proof is checked by the Lean kernel. ∎

**Theorem 5.4** (Minimal nonzero vector bounds the maximal slope) LEAN ✓

Let $L \leq E$ be a nonzero discrete lattice. If every nonzero vector $x \in L$ satisfies $\|x\| \geq r > 0$, then

$$\mu_{\max}(L) \leq -\ln r + \frac{1}{2}\ln\big(\mathrm{rk}(L)\big).$$

LEAN › `Formalism.F.muMaxLeMin`

**Proof.** Apply Theorem 5.2 to any nonzero sublattice $N \leq L$ to obtain $r \leq \|v\| \leq \sqrt{\mathrm{rk}(N)}\ \mathrm{covol}(N)^{1/\mathrm{rk}(N)}$; taking logarithms and the supremum over $N \leq L$ with $\mathrm{rk}(N) \leq \mathrm{rk}(L)$ yields the bound. The full proof is checked by the Lean kernel. ∎

See the two-way duality between slopes and vectors? Bost's slope inequality (`SlopeInequality` in `LatticeSlopes/Statement.lean`) applies the exact same roof picture across any injective integer-linear map $\varphi : L \to M$ with operator norm bound $C$:

$$\mu(L) \leq \mu_{\max}(M) + \ln C.$$

If your integral structure has no hidden ultra-steep sublattice spikes, the arithmetic height along the entire lift chain stays strictly under control!

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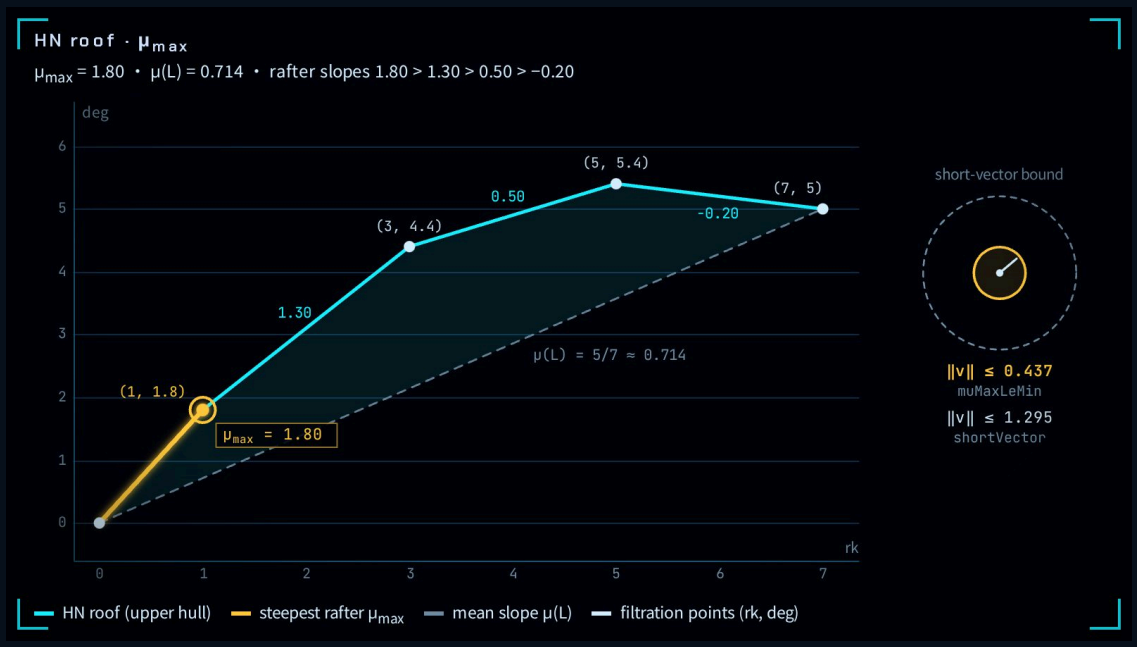



**Figure 6** THETA//SLOPE · Theta chain budget · a chain of √X mirrors stalls at 2/3 until a glued family flattens the roof

**Interactive Harder–Narasimhan concave roof:** Upper concave hull through $(0,0),(1,1.8),(3,4.4),(5,5.4),(7,5.0)$ comparing the steepest first rafter $\mu_{\max}=1.8$ (amber) against the dashed mean slope $5/7\approx 0.714$. In the full lab, dragging $(1,h_1)$ below $4.4/3$ snaps the first roof chord cleanly to $(3,4.4)$.
Interactive version:
http://rh.trureturing.org/riemann-hypothesis/viz/theta-chain-budget/

## 5.2 Vaulting $2/3$: Gluing an Integral Structure Across Hecke Characters

Now return to the crucial question: **if one Hecke character alone stalls at $2/3$, what cost does "gluing across Hecke characters" save?**

Think of shipping groceries from $K$ stalls at a market. If each stall forces you to rent a separate shipping crate, freight costs scale by $K$ and choke your budget right at $2/3$. But if you find a **common integral gcd lattice** holding all $K$ deliveries inside a single compact matrix—and check that the columns corresponding to different characters are **almost orthogonal**—then there is no collinear collapse, the Gram determinant stays cleanly near the product of column lengths, and the shared structure spreads the cost across the entire family!

In the architecture of the proof (Steps 15–16 of the author's summary), that gluing works as a two-punch combination:

1. **Assemble a shared integral structure across Hecke characters** (an extra averaging / "gcd" over characters), backed by a relative trace formula via automorphy on a

sixfold cover of $\mathrm{GL}_5$ and Cauchy–Schwarz against the norm of a projected Poincaré series.

2. **Prove approximate orthogonality of the columns** of that glued integral matrix. With no collapsed directions, the maximal slope $\mu_{\max}$ stays tame, driving the threshold abscissa cleanly past $\frac{2}{3}$ all the way down into $\frac{1}{2} + \varepsilon$!

> One thin stick snaps two-thirds of the way across the river; weave nearly orthogonal vectors into a single integral lattice, and the bridge holds all the way to one-half.

Yet two final pockets of resistance still lurk along the road to every Dirichlet character: accidental arithmetic-progression resonances inside the Iwasawa tower levels, and stubborn Galois cohomology classes that refuse to vanish by twisting alone.

In the next chapter, we bring in Behrend's high-dimensional integer sphere shell and a strong weakening of Vandiver's conjecture to close the net once and for all.

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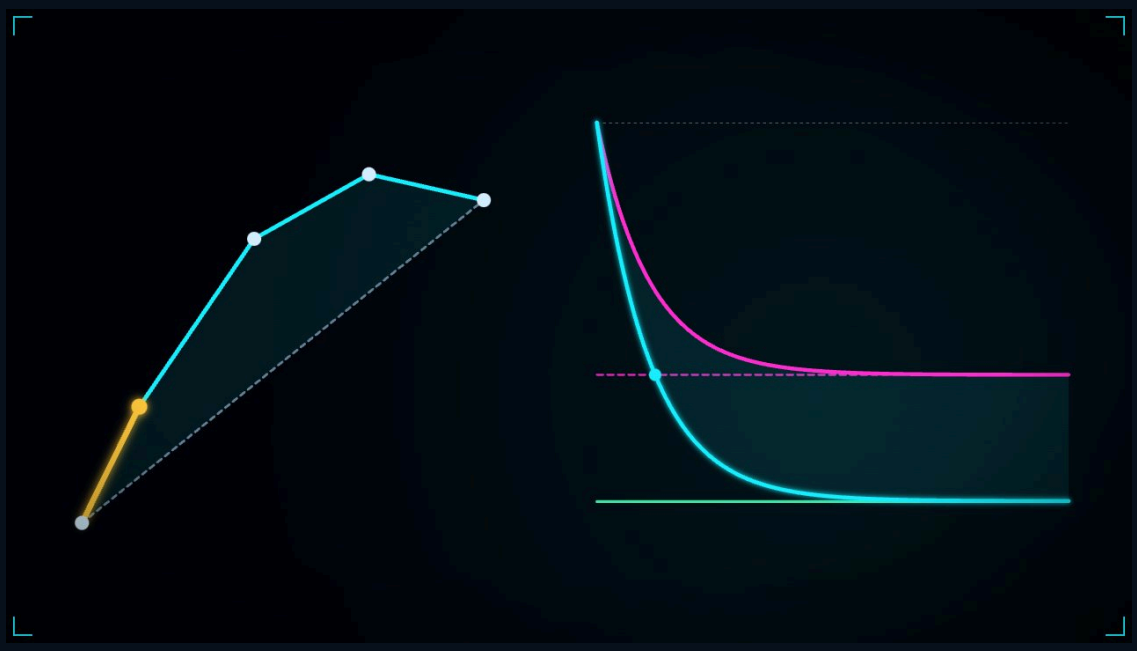


**Figure 7** THETA//SLOPE · Theta chain budget · a chain of √X mirrors stalls at 2/3 until a glued family flattens the roof

**The full theta-chain and lattice-slope lab:** Switch between the mirror-chain budget and the interactive Harder–Narasimhan roof, adjust depth ratio $m/\sqrt{X}$, character gluing, and first-rafter degree $h_1$, and verify how orthogonal gluing pulls the threshold below $2/3$.
Interactive version:
http://rh.trureturing.org/riemann-hypothesis/viz/theta-chain-budget/

§ 06

# Chapter 6 Behrend Towers and Closing the Net: Taming Obstructions Across All Characters

> Place integers on a high-dimensional sphere shell: because a sphere curves strictly outward, no three distinct points on the shell can ever form an evenly spaced straight line—yet the heaviest shell still captures a remarkably dense crowd of numbers. Combine that resonance-free selection with local-global Galois control, and the net tightens over every Dirichlet character at once.

By the end of Chapter 5, we had the engine to shatter the $2/3$ ceiling: a $\sqrt{X}$-step theta-lift chain glued across families of Hecke characters. Yet turning that engine loose on **every modulus $N$ and every Dirichlet character $\chi$** still faces two last pockets of resistance:

1. **Arithmetic-progression resonances inside tower levels**: when choosing Iwasawa tower levels or parameter shifts, accidental three-term arithmetic progressions $s_2 - s_1 = s_3 - s_2$ (or linear relations $\alpha s_1 + \beta s_2 + \gamma s_3 = 0$) let error waves line up in phase and reinforce one another.
2. **Stubborn Galois cohomology classes**: classes in $H^1(\mathfrak{o}_E[1/\ell], \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))$ that refuse to vanish under character twisting—requiring a substantial weakening of Vandiver's conjecture and sharp local-global Langlands compatibility to keep under polynomial control.

## 6.1 Why an Integer Sphere Shell Never Holds a 3-Term Arithmetic Progression

Start with a pocket-sized gem discovered by Felix Behrend in 1946: **how to pack a dense crowd of integers while forbidding any three-term arithmetic progression $a, b, c$ ($b = \frac{a+c}{2}$).**

Picture the $5 \times 5 \times 5$ integer cube $\{0, 1, 2, 3, 4\}^3$ containing $125$ lattice points. Group those points by squared radius $R = x_1^2 + x_2^2 + x_3^2$. Since $0 \le R \le 3 \cdot 4^2 = 48$ and not every integer is a sum of three squares, the $125$ points land on only $32$ distinct spherical shells!

By the plain pigeonhole principle, some shell must hold at least $\lceil 125/32 \rceil = 4$ points. In fact, computing all shells reveals a champion shell at $R = 17$ carrying **9 points**:

$$(0,1,4),\ (0,4,1),\ (1,0,4),\ (1,4,0),\ (2,2,3),\ (2,3,2),\ (3,2,2),\ (4,0,1),\ (4,1,0).$$

Now check by pure geometry: **can any three points $A, B, C$ on that shell satisfy $B = \frac{A+C}{2}$?**

Impossible! A sphere is **strictly convex**. Take any two distinct points $A \neq C$ of squared length $R$. Their midpoint $B = \frac{A+C}{2}$ sits **strictly inside** the sphere:

$$\|B\|^2 = \frac{\|A\|^2 + \|C\|^2}{2} - \frac{\|A-C\|^2}{4} = R - \frac{\|A-C\|^2}{4} < R.$$

So $B$ can never touch the shell of radius $\sqrt{R}$! Encode each 3D vector $(x_1, x_2, x_3)$ in base $B > 8$ as a single integer $x_1 + x_2 B + x_3 B^2$ (no carries mix between digits), and the sphere shell instantly becomes an **integer set with zero 3-term arithmetic progressions**—and in high dimension $d$, its size hits the astonishing density $N \exp(-C\sqrt{\ln N})$, dodging resonances while barely sacrificing capacity!

VIZ

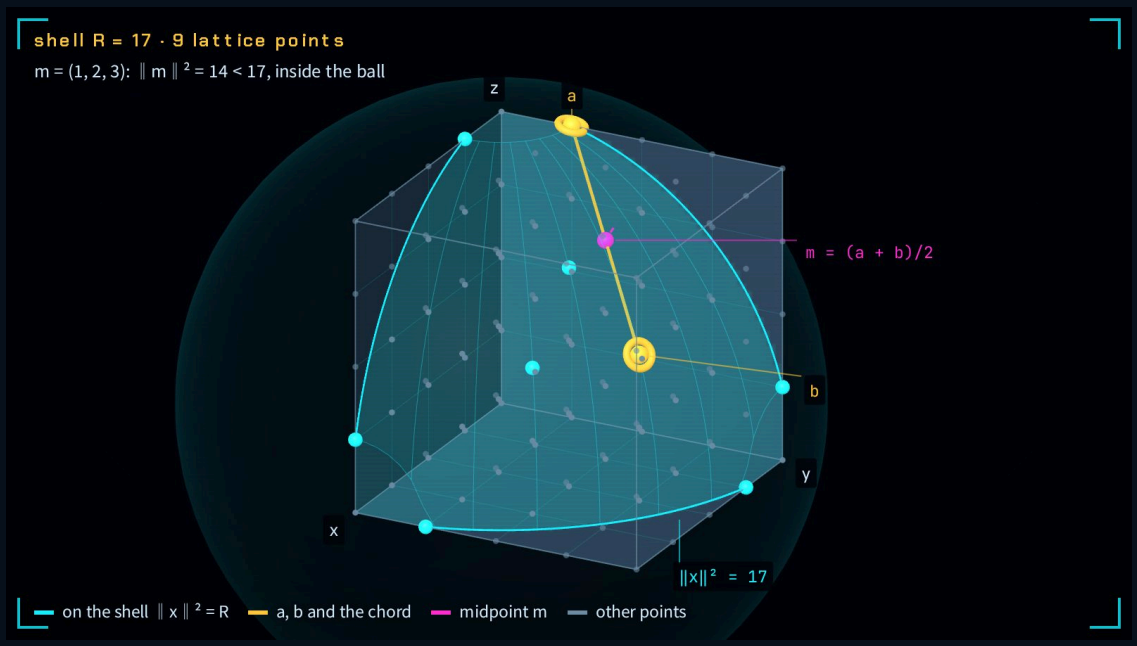



**Figure 8** SPHERE//SHELL · Behrend sphere · no three-term progression on an integer shell

**The 3D Behrend sphere excludes 3-term arithmetic progressions:** On the translucent glass sphere $R = 17$ inside $\{0, \ldots, 4\}^3$ sit exactly 9 lattice beads (cyan). Picking amber endpoints $a = (0, 1, 4)$ and $b = (2, 3, 2)$ sends their chord midpoint $m = (1, 2, 3)$ (magenta) strictly inside the sphere at $\|m\|^2 = 14 < 17$—guaranteeing zero 3-term arithmetic progressions.
Interactive version:
http://rh.trureturing.org/riemann-hypothesis/viz/behrend-sphere/

Using such thick Behrend sets together with probabilistic derandomization (`select`) and Galois stabilizer rigidity (`trivStab`), the proof selects clean levels while keeping finite stabilizers trivial:

**Theorem 6.1** (Deterministic subset selection under binomial variance) LEAN ✓

Let $X$ be a finite set, $\delta \in [0,1]$, $Y : \mathcal{P}(X) \to \mathbb{R}_{\geq 0}$ a non-negative cost functional, and $\lambda > 0$ a margin satisfying $2|X|\delta(1-\delta) < \lambda^2$. Then there exists a subset $T \subseteq X$ simultaneously satisfying

$$\big||T| - \delta|X|\big| \leq \lambda \qquad \text{and} \qquad Y(T) \leq 2\,\mathbb{E}_{X,\delta}[Y].$$

LEAN › Formalism.BehrendSets.Final.select

*Proof.* Equip $\mathcal{P}(X)$ with the $\delta$-weighted Bernoulli product measure. By Chebyshev's inequality the probability of $|T|$ deviating by more than $\lambda$ from $\delta|X|$ is strictly less than $1/2$, while by Markov's inequality the probability that $Y(T) > 2\,\mathbb{E}[Y]$ is at most $1/2$. Their sum is strictly under $1$, forcing a surviving subset $T$ satisfying both bounds. The full proof is checked by the Lean kernel. ∎

**Theorem 6.2** (Triviality of large prime-power subgroups in $\mathrm{GL}_k(E)$) LEAN ✓

Let $E$ be a number field, $k \in \mathbb{N}$, and $g$ a prime such that $[E : \mathbb{Q}] \cdot k < g - 1$. Then every finite $g$-subgroup $\Gamma \leq \mathrm{GL}_k(E)$ is trivial:

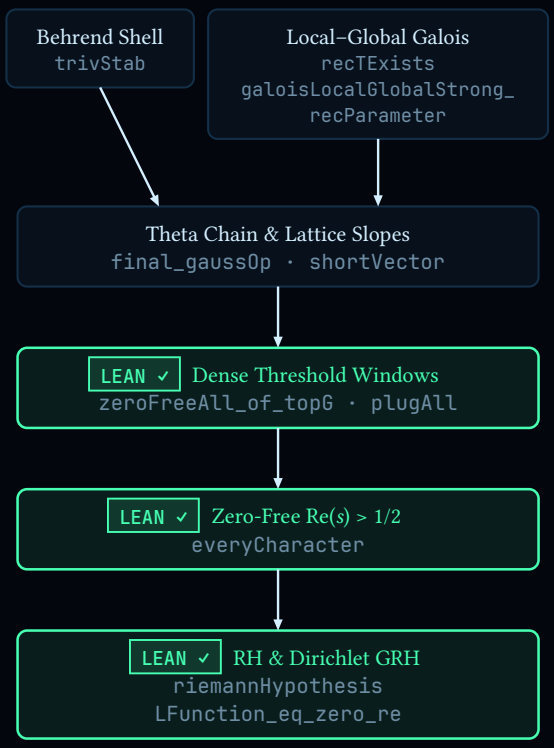
$$\Gamma = \{1\}.$$

LEAN › Formalism.BehrendSets.FinalE.trivStab

*Proof.* If $\Gamma$ were nontrivial, it would contain an element $\gamma$ of exact prime order $g$, forcing a primitive $g$-th root of unity into an extension of $E$ of degree at most $k$. Comparing against $[\mathbb{Q}(\zeta_g) : \mathbb{Q}] = g - 1$ contradicts $[E : \mathbb{Q}] \cdot k < g - 1$. The full proof is checked by the Lean kernel. ∎

## 6.2 From Local Langlands to Closing the Net

Now watch the final steel girders lock into place at the top of `src/Development/AutomorphicArithmetic/`:




**Top-level dependency spine of the proof.** Behrend selection and stabilizer rigidity (`trivStab`), local Langlands central-unit compatibility (`recTExists`), and local-global Galois integral structures (`galoisLocalGlobalStrong_recParameter`) feed the small-class-number Selmer bound (`corGK`) and dense threshold window engine (`zeroFreeAll_of_topG`), converging through `plugAll` and `everyCharacter` to right-half-plane nonvanishing and the Riemann Hypothesis.

Six tightly interlocked Lean theorems seal the entire chain:

1. **Local Langlands and tame type laws**: `RecTExists.lean` constructs the local Langlands map `rc` at every finite place, matching dimension, parabolic segments, supercuspidals, central characters, duality, and tame type polynomials.

**Theorem 6.3** (Existence of local Langlands correspondence with type law) LEAN ✓

The full local Langlands and tame-type package `UpperTheory.Zero.LLC.RecTExists` holds, furnishing a correspondence `rc` preserving dimension, one-dimensional characters, segment decompositions, cuspidal irreducibility, duality, injectivity, depth, and `TypeLaw`.

LEAN › `UpperTheory.Zero.LLC.recTExists`

*Proof.* Starts from local Artin reciprocity and cuspidal correspondence, extends to general irreducible representations via `exists_isLExt`, and verifies central character norms, twists, depth preservation, and inertial characteristic polynomials. The full proof is checked by the Lean kernel. ∎

2. **Local-global Galois integral bridge**: `RowSrc.lean` feeds the theta-lift arithmetic parameters into the global integral structure axiom `Row0b`.

**Theorem 6.4** (Strong local-global Galois integral structure from theta lifts) LEAN ✓

The strong local-global integral structure predicate `UpperTheory.GalStar.Row0b recT` holds.

LEAN › `UpperTheory.Gal.Int.galoisLocalGlobalStrong_recParameter`

Proof. Applies `galoisRepresentations_of_thetaLift` directly to the theta-lift construction `tLiftK`. The full proof is checked by the Lean kernel. ∎

3. **Small class numbers and the weakened Vandiver bound**: `CorGKOf.lean` proves `CorGK` —controlling total log-weighted Bloch–Kato Selmer lengths across all primes $l$ by a polynomial bound $2c_5\varphi(H.\mathrm{index})(N(e+1))^A(\ln p + \ln p')$, so residual cohomology classes cannot derail the gluing budget.

**Theorem 6.5** (Global Selmer-length bound across primes) LEAN ✓

The global Selmer-length bound `UpperTheory.V.U3.CorGK` holds across number fields, level subgroups $H$, and finite prime sets $L_s$, bounding $\sum_{l\in L_s}\ln l\sum_{\psi}V_n$ by $2c_5\varphi(H.\mathrm{index})(N(e+1))^A(\ln p+\ln p')$.

LEAN › `UpperTheory.V.SmallClassNumbers.corGK`

Proof. Combines patched local-global cohomology inputs (`litN`) with Kato's Euler-system bounds, adjoint lifting, and admissible type-I reductions to deduce `PropGK`, then pairs two auxiliary good primes $p \neq p'$ to eliminate the single-prime exclusion. The full proof is checked by the Lean kernel. ∎

4. **Dense threshold windows across every primitive and imprimitive character**: `ZeroFreeAllOfTopG.lean` and `EveryCharacter.lean` assemble Behrend-filtered family windows, slope bounds, and local-global coherence first into `ZeroFreeAll` (all primitive characters) and then into `EveryCharacter` (all moduli and characters, imprimitive included).

**Theorem 6.6** (Dense threshold connectivity implies primitive right-half-plane nonvanishing) LEAN ✓

For any step ratio $\lambda > 1$, the dense threshold connectivity property `TopG` together with host window existence `HostExW la (GChi ψ).req` for every primitive character $\psi$ implies `ZeroFreeAll`.

LEAN › `UpperTheory.Top.zeroFreeAll_of_topG`

Proof. Extracts Landau-threshold lower bounds (`eFamC_of_cls`) and height bounds (`hypV_of_cls`) along connected family classes and applies `zeroFreeOn_of_tiedWin_cyc` across comparison scales. The full proof is checked by the Lean kernel. ∎

**Theorem 6.7** (Assembly of right-half-plane nonvanishing for all Dirichlet characters) LEAN ✓

Given Green-type bounds `hG`, weakened Vandiver window control `hV : VWin`, and closed link design existence `h`, `UpperTheory.EveryCharacter` holds: $L(\chi, s) \neq 0$ for every modulus $N$, every character $\chi$, and every $\mathrm{Re}(s) > \frac{1}{2}$ ($\chi \neq 1$ or $s \neq 1$).

LEAN › UpperTheory.Top.plugAll

*Proof.* Combines Theorem 6.6 with `shapeWin` and `supplyChi` to establish `ZeroFreeAll` (`plugWin`), then applies `imprimitive` using the fact from Chapter 1 that extra Euler factors cannot vanish on $\mathrm{Re}(s) > 0$. The full proof is checked by the Lean kernel. ∎

5. **Unconditional finale:** `MainConclusion.lean` discharges every hypothesis unconditionally:

**Theorem 6.8** (Unconditional right-half-plane nonvanishing for all Dirichlet $L$-functions) LEAN ✓

Unconditionally, for every $N \geq 1$, every Dirichlet character $\chi$ modulo $N$, and every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$ and ($\chi \neq 1$ or $s \neq 1$), $L(\chi, s) \neq 0$.

LEAN › UpperTheory.FinalTheorems.everyCharacter

*Proof.* Feeds the unconditional theorems Theorem 6.4 and Theorem 6.3 into `every_character_of_literature_of_vWin`, discharging `UpperTheory.EveryCharacter` with zero unproved assumptions. The full proof is checked by the Lean kernel. ∎

From the functional-equation mirror at $\mathrm{Re}(s) = \frac{1}{2}$, down into Landau's one-sided prime ledger, across Stepanov's Fock/Weyl multiplicity, along the $\sqrt{X}$ unitary theta chain and Seesaw flip, through Harder–Narasimhan slopes and glued Hecke structures past $2/3$, and finally through Behrend's sphere shell and global Selmer bounds—the proof's spine stands end to end.

Now ask the toughest engineering and scientific question left: **how can any mathematician trust a 17-million-line uncommented Lean repository without taking anyone's word for it?**

Chapter 7 opens the audit bay.

VIZ

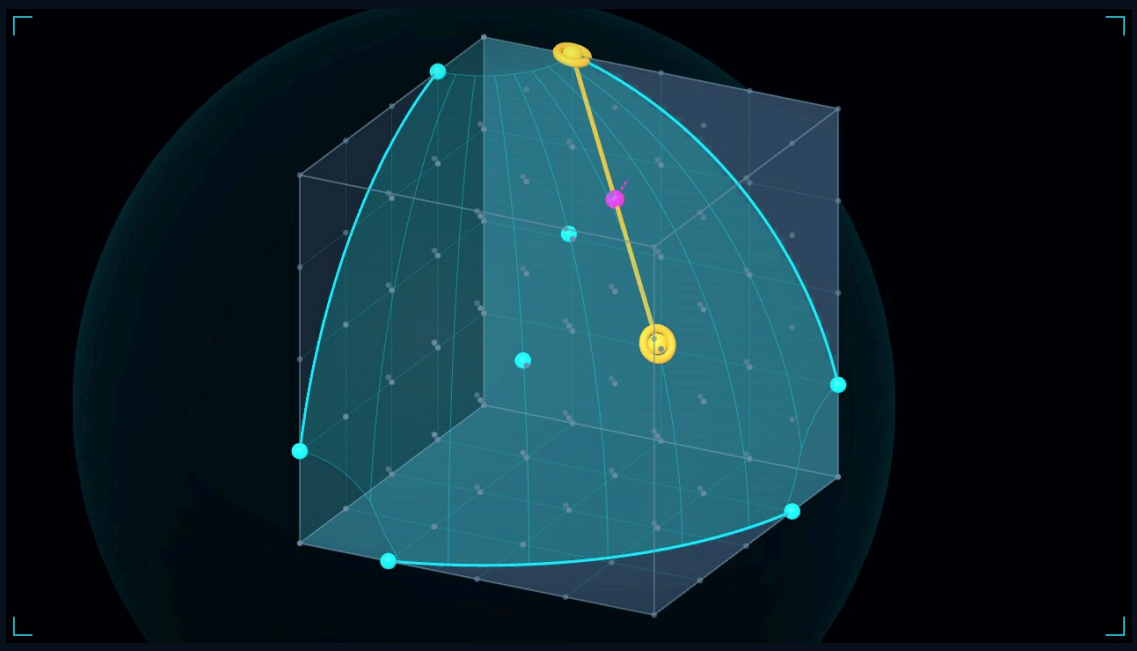


**Figure 9** SPHERE//SHELL · Behrend sphere · no three-term progression on an integer shell

**The full Behrend sphere lab:** Rotate the 3D glass sphere and lattice cube $\{0, \ldots, M-1\}^3$, vary the cube edge $M \in [3, 6]$ and squared-radius shell $R$, pick any two shell beads to inspect their interior midpoint drop, and explore the pigeonhole shell histogram below ($|S_{17}| = 9$ at $M = 5$).
Interactive version:
http://rh.trureturing.org/riemann-hypothesis/viz/behrend-sphere/

§ 07

# Chapter 7 Anatomy of 17 Million Lines: Independent Kernel Replay and Mathematical Honesty

> Anyone who hands you a 175,187-file, 17.37-million-line manuscript with zero comments and says “trust me” might as well be selling indulgences. Fortunately, formal verification lives by a colder, cleaner rule: **do not trust the author’s word—replay the compiled output inside an empty kernel.**

If this proof were a ten-thousand-page PDF, a human referee board might need twenty years just to turn every page—and still could not swear that page 4,812 did not hide a dropped inequality or an unstated exception.

Before Lean 4’s kernel, massive length buys zero immunity. Every single line must elaborate through the same rigid type checker. Even one hidden `sorry`, one custom `axiom`, or one sneaky rewrite of `RiemannHypothesis` would be caught instantly by a 27-line challenge file.

## 7.1 The 27-Line Challenge Contract: How Verification Prevents Cheating

Before opening `the-omega-institute/riemann-hypothesis`, let us keep its provenance crystal clear: the 17.37-million-line Lean 4 proof source for RH and Dirichlet GRH was authored and released by **Levent Alpöge** (`alpo.ge/remark.pdf`); **Deicyde** then created the upstream verification mirror `Deicyde/riemann-hypothesis`, adding a 27-line Mathlib-only challenge contract `src/Challenge.lean` (importing **only** official Mathlib `v4.33.1` at commit `0df444a360ea`) and Comparator harness around the unmodified source; and `the-omega-institute/riemann-hypothesis` is The Omega Institute’s **fork** of `Deicyde/riemann-hypothesis`, keeping every proof file untouched while adding the 8,490-shard public CI builder and `./verify` local replay script.

That contract does not care whether the proof inside uses unitary groups, CM elliptic curves, or moonbeams. It asks five blunt questions:

1. Does the type of `Main.riemannHypothesis` match Mathlib’s native `RiemannHypothesis` **definitionally and verbatim**?
2. Do the other four headline theorems in `Main.lean` match the short Mathlib-only definitions copied into `Statements.lean`?

3. Across all 175,187 `.lean` files, is there zero occurrence of `sorry`, `admit`, `axiom`, `native_decide`, `unsafe`, `implemented_by`, `opaque`, `debug.*`, or `skipKernelTC`?
4. Do all nine dependencies in `lake-manifest.json` (`mathlib`, `batteries`, `aesop`, `Qq`, and their companions) point to untouched official upstream commits?
5. Inside an offline, read-only, networkless sandbox, do Lean's own kernel and independent standalone checkers (`leanchecker-paranoid`, `nanoda`, `con-leche`, `con-ron`, and `lean4lean`) replay the export from scratch and confirm that the axiom closure uses nothing outside `{propext, Classical.choice, Quot.sound}`?

> You may build a labyrinth seventeen million lines deep, yet the gatekeeper asks only two questions: did you solve the exact problem everyone agreed on, and did you stand on anything beyond standard axioms?

Here are the four headline assembly theorems collected in `Supplements/Kernels/MainTheorems.lean`, which feed straight into `Main.lean`:

**Theorem 7.1** (Kernel bundle: nonvanishing for every Dirichlet character) LEAN ✓

`UpperTheory.EveryCharacter` holds: for every modulus $N \geq 1$, every Dirichlet character $\chi$ modulo $N$, and every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$ ($\chi \neq 1$ or $s \neq 1$), $L(\chi, s) \neq 0$.

LEAN › `allChi`

*Proof.* In `MainTheorems.lean`, directly invokes Theorem 6.8. The full proof is checked by the Lean kernel. ∎

**Theorem 7.2** (Kernel bundle: the Riemann Hypothesis) LEAN ✓

Mathlib's `RiemannHypothesis` holds.

LEAN › `rhOfAll`

*Proof.* Feeds Theorem 7.1 into Theorem 1.7 via `Formalism.VW.rh_of_every`. The full proof is checked by the Lean kernel. ∎

**Theorem 7.3** (Kernel bundle: critical line for primitive characters) LEAN ✓

For every modulus $N \geq 1$, every primitive Dirichlet character $\chi$ modulo $N$, and every $s \in \mathbb{C}$ with $0 < \mathrm{Re}(s) < 1$ and $L(\chi, s) = 0$, $\mathrm{Re}(s) = \frac{1}{2}$.

LEAN › `critAll`

*Proof.* Feeds Theorem 7.1 into the primitive reflection theorem Theorem 1.11. The full proof is checked by the Lean kernel. ∎

**Theorem 7.4** (Kernel bundle: half-plane, strip, and zeta conjunction) LEAN ✓

The conjunction `Claim.HalfPlane ∧ Claim.Strip ∧ Claim.Zeta` holds: every Dirichlet $L$-function is zero-free on $\mathrm{Re}(s) > \frac{1}{2}$ (away from $\chi = 1, s = 1$), every critical-strip zero of every Dirichlet $L$-function has $\mathrm{Re}(s) = \frac{1}{2}$, and every nontrivial zero of $\zeta(s)$ has $\mathrm{Re}(s) = \frac{1}{2}$.

```
LEAN › claimAll
```

**Proof.** Feeds Theorem 7.1 into Theorem 1.12 and its specialization `Claim.zeta` to form the three-way conjunction. The full proof is checked by the Lean kernel. ∎

## 7.2 8,490 Content-Addressed Shards and Public CI

Here is the practical bottleneck: if someone announces a formal proof of the Riemann Hypothesis, yet sets the price of admission at "rent a 503 GB server and burn ten uninterrupted hours," independent checking stays locked behind a few private doors.

That single-machine cost is real. Compiling the full `Main` import closure from scratch on a 48-core, 503 GB Linux workstation takes roughly 7 to 10 hours, and Lake writes about 6 MB of intermediate `.setup.json` metadata per module—meaning nearly **1 TB** of transient files across 175,187 modules. Here is how those files break down:

| Library Layer | Files | Lines of Lean | Architectural Role |
|---|---|---|---|
| `Development` | 141,928 | 12,232,663 | Automorphic arithmetic, abelian varieties, character boxes, final assembly |
| `Formalism` | 18,254 | 3,200,748 | Landau reduction, lattice slopes, Fock/Weyl, theta lifts, Behrend tools |
| `GroundTheory` | 8,507 | 1,119,138 | Arithmetic geometry, local/global fields, $p$-adic Hodge and cohomology base |
| `ReciprocityTheory` | 3,845 | 563,016 | Chebotarev, class field theory, adele approximation, local characters |
| `CompanionTheory` | 2,027 | 194,218 | Real discrete series, Harish-Chandra modules, spectral geometry |
| `PositivityTheory` | 449 | 55,130 | Automorphic positivity and published theorem interfaces |
| `Lemmata` | 167 | 9,232 | Determinant monotonicity, norm bounds, auxiliary algebraic lemmas |
| Top entry & supplements | 10 | 70 | `Main.lean`, `Statements.lean`, `BuildAid`, kernel summary bundles |

To turn "only a cluster can build this" into "anyone can watch CI compile it piece by piece and replay the result on an ordinary PC," `ci/plan.py` in `the-omega-institute/riemann-hypothesis` slices the true `import` DAG of `Main` into **8,490 bounded shards**:

- **Hard size caps and massive fan-out**: Each shard holds at most **400 modules** and at most **2.5 MB** of Lean source across **36 dependency levels**. Once the **17 foundation shards** (`Statements`, `BuildAid`, and the base packages) land, they immediately unlock **529 independent jobs** that run in parallel.
- **Content-addressed keys (no wasted rebuilds)**: Every shard key is a SHA-256 hash of four ingredients—the toolchain and manifest config (`lean-toolchain`, `lake-manifest.json`, `lakefile.toml`), the exact Git blob hashes of the shard's modules, and the sorted keys of its prerequisite shards. Edit one leaf file, and only its ancestor cone to `Main` changes key; every untouched branch keeps its cache hit.
- **Critical-path CI scheduling into 16 public Releases**: The public GitHub Actions workflow (`lean-parts.yml`) checks ready work every minute, prioritizes the shards on the longest remaining paths to `Main`, and publishes `zstd`-compressed archives with SHA-256 checksums across 16 hexadecimal Pre-release buckets (`lean-parts-v1-0` through `lean-parts-v1-f`, with Actions-artifact fallback). Public CI has already built and published the vast majority of the shards and is actively pushing the final layers toward `Main` and the final `build-<commit>` release.

The interactive diagram below lays out how those 8,490 content-addressed shards unlock level by level toward `Main.lean`:

VIZ




**Figure 10** PROOF//REPLAY · Proof replay · 8,490 content-addressed shards and a zero-trust local kernel replay

**8,490 content-addressed shards across 36 levels.** Scrub the CI wave slider or click any shard node to see how the 17 foundation shards on the left fan out into 529 parallel jobs, how content keys isolate edits, and how the frontier climbs toward `Main`.
Interactive version:
http://rh.trureturing.org/riemann-hypothesis/viz/proof-replay/

## 7.3 Anyone Can Verify It: Four Commands and Zero-Trust Replay

> "Wait—if you download pre-compiled binaries from public CI, aren't you just trusting the CI runners?" Quite the opposite: buying bricks off someone else's kiln does not mean trusting their sales pitch, as long as you put every brick through your own press when it arrives.

That distinction is the secret weapon of the verification workflow: it cleanly separates *saving compilation hours* from *granting trust*. You need **no GitHub account**, and you do not have to burn ten hours on 48 cores. On any x86-64 Linux machine with `git`, Python `3.12+`, `curl`, `zstd`, and `elan` (which picks up the pinned Lean `v4.33.1`), four commands run the check locally (even on macOS, `--status` immediately inspects live Release coverage):

```sh
git clone --depth 1 https://github.com/the-omega-institute/riemann-hypothesis.git
cd riemann-hypothesis
./verify --status Main
./verify Main
```

Two helper commands let you test the pipeline right now or finish any straggling tail yourself:

```sh
./verify BuildAid.D1           # Test the full download → SHA-256 → fresh replay path in seconds
./verify --build-missing Main  # Restore published CI shards, then compile any unpublished tail locally
```

Why is downloading `.olean` shards zero-trust? Open up `./verify` and trace what happens after download:

1. **SHA-256 manifest verification**: Every downloaded shard archive is checked against its GitHub Release digest and the source tree's exact shard key before unpacking.
2. **Fresh empty-kernel replay**: The script does **not** rubber-stamp `.olean` headers. It runs `lake env leanchecker --fresh --verbose Main`. That `--fresh` flag spins up a **completely empty Lean 4 kernel** and re-type-checks every single declaration across the import closure from scratch on your CPU. Swap one proof term, forge one binary byte, or hide a single `sorry`, and the fresh kernel refuses that declaration on the spot.
3. **Definition & axiom audit via `Audit.lean`**: Once `leanchecker --fresh` accepts `Main`, `ci/Audit.lean` verifies definitionally that `riemannHypothesis` matches Mathlib's native `RiemannHypothesis`, audits `#print axioms` on all five headline theorems against `{propext, Classical.choice, Quot.sound}`, and writes the report to `.lake/parts/axioms.txt`.

Use the fault-injector experiment below to play attacker: flip between a clean build, a forged shard, a smuggled axiom, and a mutated theorem statement to watch which gate catches each lie red-handed:

VIZ

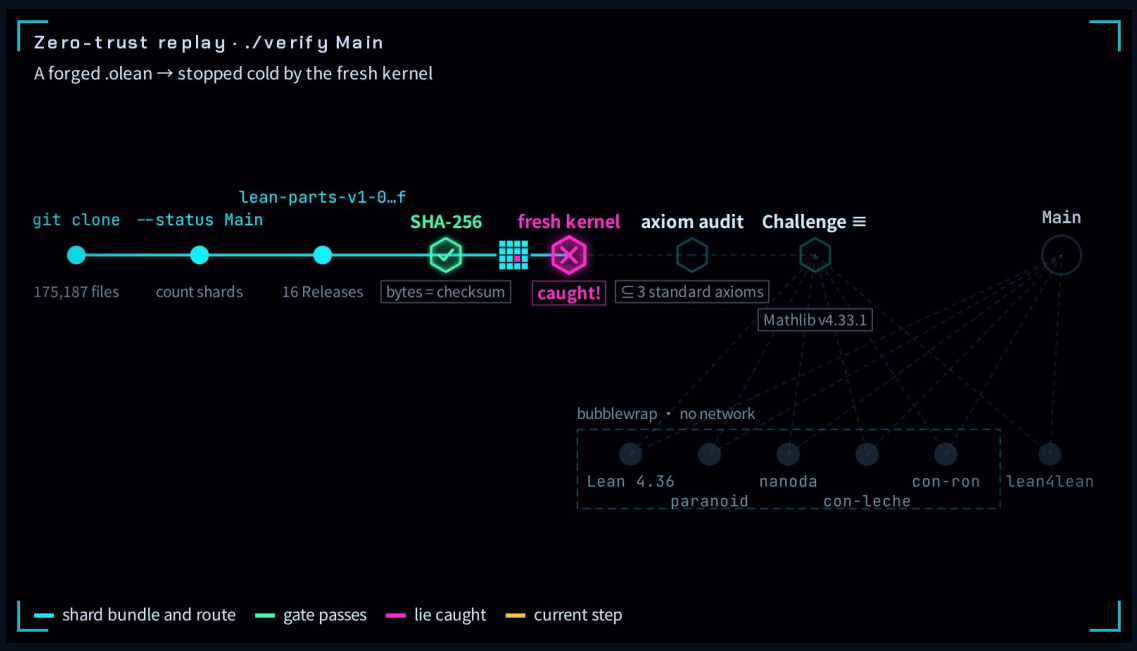



**Figure 11** PROOF//REPLAY · Proof replay · 8,490 content-addressed shards and a zero-trust local kernel replay

**Zero-trust replay: try to cheat.** Switch among clean replay, a forged `.olean` shard, a hidden `sorry`/axiom, and an altered target statement to see why downloading CI artifacts never asks for blind trust.
Interactive version:
http://rh.trureturing.org/riemann-hypothesis/viz/proof-replay/

Beyond the four-line `./verify` workflow, independent auditors also configured the full **six-checker Comparator pipeline** documented in the repository `README.md` and `verification/NOTES.md`: inside an offline `podman` build and `bubblewrap` sandbox, `lean4export` writes plain S-expression exports for `Challenge` and `Main` (~70–75 GB projected for `Main`), which Lean `v4.36.0` `lake comparator` then judges with five independent checkers (**Lean 4.36 kernel**, **`leanchecker-paranoid`**, Rust **`nanoda`**, **`con-leche`**, and **`con-ron`** configured via `comparator.json`). Because Lean `4.36` changed `Decidable` in core, the sixth checker—**`lean4lean`**—is built directly against pinned Lean `v4.33.1` and run separately as `lake env lean4lean --fresh Main`.

Step into the full verification lab below to scrub the 36-level CI frontier, inspect content-addressed invalidation cones, test the fault injector, and cross-check all five locked `Main` theorem signatures:

VIZ

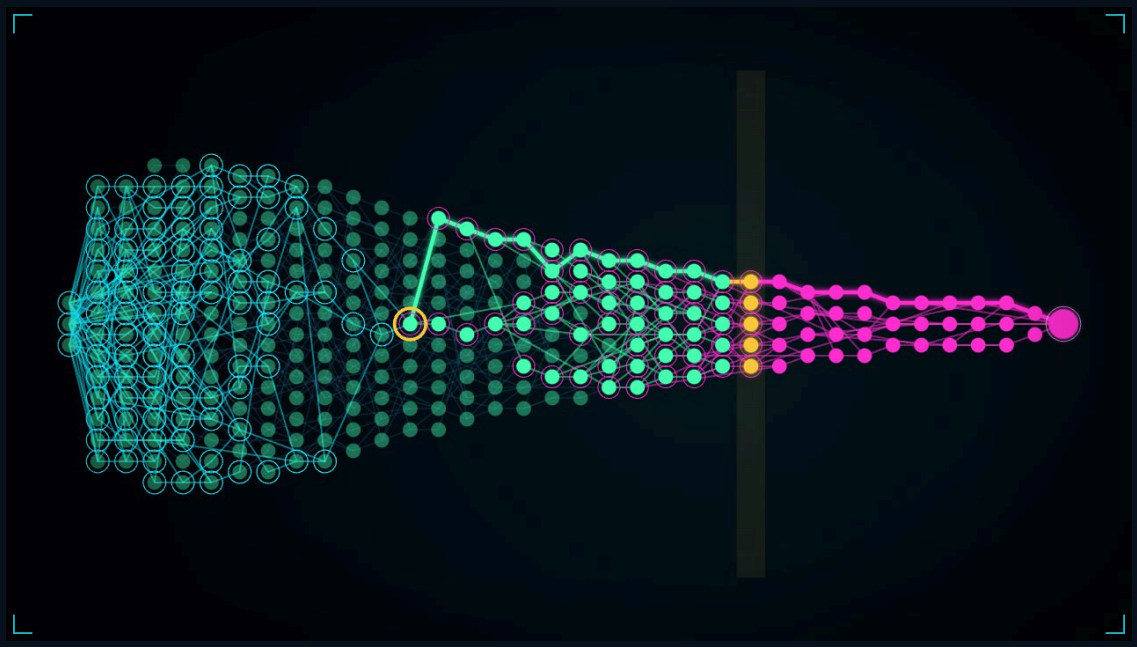


**Figure 12** PROOF//REPLAY · Proof replay · 8,490 content-addressed shards and a zero-trust local kernel replay

**Full Verification Lab · From 8,490 public CI shards to four-command `./verify` and six independent checkers.** Click shard nodes in Scene 1 to inspect ancestor invalidation; switch to Scene 2 to test every tamper vector.
Interactive version:
http://rh.trureturing.org/riemann-hypothesis/viz/proof-replay/

For a two-minute guided walkthrough of the whole verification pipeline—from the 27-line challenge contract to public CI shards and zero-trust local replay—watch the short film below.

FILM




**Figure 13** Film · Verifying 17 Million Lines of Lean · 2:15

**Verifying 17 Million Lines of Lean.** A two-minute tour of how public CI shards the proof into 8,490 content-addressed pieces, how anyone can verify `Main` with four commands and zero trust via `leanchecker --fresh`, and why six independent checkers close off cheating.
Film:
http://rh.trureturing.org/riemann-hypothesis/films/#verifying-the-riemann-proof

## 7.4 Honest Verification Boundaries

By the standards of strict scientific honesty recorded in `verification/NOTES.md`, keep three objective boundaries in full view:

1. **Multi-terabyte full Comparator replay is a long-horizon run**: the static source scan (zero `sorry`/`axiom`/`native_decide`/`unsafe`/`opaque`), the nine official upstream package pins including Mathlib `v4.33.1`, and the Mathlib-scale five-checker Comparator smoke test (54,461 declarations) all passed; public CI is steadily compiling the 8,490 shards toward `Main`, while the full 70+ GB multi-checker Comparator run executes as a standalone multi-hour offline pipeline.
2. **Scope of this book's lightweight source lock**: in `lean-source` mode, `./tbook check` verifies exact file paths, line numbers, signature digests, and direct freedom from `sorry`/`admit` in every cited theorem; replaying the full transitive import closure and axiom footprint requires Lean compilation and kernel replay (`./verify Main` or Comparator) on Linux.
3. **Shared trust base**: even when every independent checker agrees, human and machine verification alike still rest on three foundational assumptions—the soundness of Lean's type theory and kernel, the fidelity of Mathlib `v4.33.1`'s definitions

of `RiemannHypothesis` and `DirichletCharacter.LFunction`, and the uncompromised hardware and sandbox executing the replay.

> A machine cannot hand you understanding on a silver platter—but it can guarantee that not half a lie hides anywhere inside seventeen million lines. True beauty never minds facing the cold iron of the checker.

In the Coda, let us step outside the finished cathedral and look toward the peaks that still stand unconquered beyond it.

# Coda The Mountains Beyond Riemann

> The first ascent of a great conjecture is rarely a paved highway; it looks more like an expedition loaded with ropes, crampons, and sixteen overlapping maps. Once the flag reaches the summit, the thrill of the next generation is watching mathematicians carve a thirty-mile ridge climb into a crystal-clear afternoon walk.

When the last checker flashes green, the world does not grind to a halt. On the contrary, the best part of the adventure begins right then.

As the author remarked candidly in the announcement note, seventeen million lines of first-generation Lean proof almost certainly hide bulky scaffolding built under night-shift pressure—detours through high $K$-groups or specific tower choices that later mathematicians will likely distill down to dozens of gleaming pages. We have seen that story play out in every era of mathematics: the first proofs by Abel, Galois, or Wiles felt vast and intricate, until twenty years later graduate textbooks turned their core ideas into three clean chapters.

Just as importantly, honesty demands drawing a bright red border around what this proof **does not** settle. Proving the Riemann Hypothesis and GRH for Dirichlet $L$-functions does not clear every problem bearing Riemann's name. Four great peaks still stand unconquered on the horizon, marked cleanly below as open conjectures and problems:

**Conjecture Coda.1** (Extended Riemann Hypothesis (ERH) for general Dedekind zeta functions) OPEN

For every algebraic number field $K$, all nontrivial zeros of its Dedekind zeta function $\zeta_K(s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$ satisfy $\mathrm{Re}(s) = \frac{1}{2}$.

As documented in `verification/NOTES.md`, the formalization proves RH for $\zeta(s)$ and GRH for all Dirichlet $L$-functions over $\mathbb{Q}$; general Dedekind zeta functions and Artin $L$-functions remain outside its proved scope.

**Conjecture Coda.2** (Grand Riemann Hypothesis for general automorphic $L$-functions) OPEN

For every cuspidal automorphic representation $\pi$ on $\mathrm{GL}_n$ over a global field (including elliptic-curve Hasse–Weil $L$-functions), every nontrivial zero of $L(s, \pi)$ in the strip $0 < \mathrm{Re}(s) < 1$ lies on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

Lifting the $\mathrm{GL}_1$ Dirichlet machinery to higher-rank automorphic $L$-functions remains one of the towering challenges of twenty-first-century number theory.

**Conjecture Coda.3** (Simplicity of zeta zeros) OPEN

Every nontrivial zero $\rho$ of $\zeta(s)$ in $0 < \mathrm{Re}(\rho) < 1$ is simple: $\zeta'(\rho) \neq 0$.

**Open problem Coda.4** (Montgomery's pair correlation and GUE spectral statistics) OPEN

Prove that as height $T \to \infty$, the normalized spacings and pair correlations of the critical-line zeros $\frac{1}{2} + i\gamma$ converge rigorously to those of eigenvalues in the Gaussian Unitary Ensemble (GUE), and clarify whether a genuine Hilbert–Pólya self-adjoint operator underlies them.

Solving a 160-year-old conjecture does not close mathematics—it turns an ancient cliff into solid ground to build upon. Standing at last on the firm bedrock of $\mathrm{Re}(s) = \frac{1}{2}$, what shines sharpest across the sea are the peaks still waiting for light.

> Never mourn the solving of a sacred riddle; every old mystery that yields its key unlocks ten wilder frontiers beyond the ridge.

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# Verification ledger

The Lean citations below link to declarations in riemann-hypothesis at the pinned commit.

Formal basis: trureturing commit `0bcc9368e418` · `riemann-hypothesis`

| Item | Title | Lean declaration | Signature digest |
|---|---|---|---|
| Theorem 1.1 | The Riemann Hypothesis | `riemannHypothesis` | `1527a6aa9b` |
| Theorem 1.2 | Nontrivial zeros of $\zeta$ lie on the critical line | `riemannZeta_eq_zero_re` | `a48cc31aae` |
| Theorem 1.3 | Right-half-plane nonvanishing for all Dirichlet $L$-functions | `LFunction_ne_zero_of_half_lt_re` | `2794402ed9` |
| Theorem 1.4 | Generalized Riemann Hypothesis for all Dirichlet $L$-functions | `LFunction_eq_zero_re` | `e02504de05` |
| Theorem 1.5 | Critical line theorem for primitive Dirichlet $L$-functions | `LFunction_eq_zero_re_of_isPrimitive` | `8faa639103` |
| Theorem 1.7 | Clearing the right half-plane implies the Riemann Hypothesis | `Formalism.VW.rh_of_right` | `a1491a3f57` |
| Proposition 1.8 | Half-plane reduction with explicit negative-integer check | `Formalism.riemannHypothesis_of_zero_free_halfPlane` | `435ac00fed` |
| Lemma 1.9 | Extra Euler factors never vanish for $\mathrm{Re}(s) > 0$ | `Claim.factor_ne_zero` | `69703f1094` |
| Lemma 1.10 | Critical-strip reflection for primitive characters | `Claim.prim_reflect` | `e19f39b4f2` |
| Theorem 1.11 | Half-plane to critical line for primitive characters | `Formalism.VW.critLine` | `a71f8d480a` |
| Theorem 1.12 | Critical-strip reduction for all Dirichlet characters | `Claim.strip` | `02b0fab135` |
| Theorem 2.1 | Lipschitz continuity and linear bound for $G_z$ | `Formalism.zetaPrimeSumErrorLipschitz` | `6436b2dbc8` |
| Theorem 2.2 | Mellin transform of generalized von Mangoldt sums | `Formalism.mellinSgen` | `720f209266` |
| Theorem 2.3 | Landau's non-negative integral lemma | `Formalism.landauLemma_of` | `0775602139` |

| Item | Title | Lean declaration | Signature digest |
|---|---|---|---|
| Theorem 2.4 | One-sided prime-sum bound implies half-plane nonvanishing | `Formalism.zetaLandau` | `f0b439e535` |
| Corollary 2.5 | Epsilon-tolerant Landau criterion | `Formalism.zetaLandauEps` | `36578f3df1` |
| Theorem 2.6 | Classical zero-free region for $\zeta(s)$ | `Formalism.ZF.zeroFreeRegion` | `2de4263445` |
| Theorem 2.7 | Siegel's lower bound for $L(1,\chi)$ | `Formalism.ZeroFreeDirichlet.siegelLower` | `25fb97aa3c` |
| Lemma 3.1 | Symmetry and strict positivity of the rational Fock pairing | `Formalism.GradedWeylDivisibility.fockQPos` | `f1b21e1715` |
| Lemma 3.2 | Fock adjointness of divided-power row operators | `Formalism.GradedWeylDivisibility.fockQRowDivAdj` | `0ff56d0850` |
| Lemma 3.3 | Divided-power closed subspaces are row-weight graded | `Formalism.GradedWeylDivisibility.rowDivClosedGraded` | `c9fed46253` |
| Theorem 3.4 | Coincidence of rational Weyl and dual Weyl modules | `Formalism.GradedWeylDivisibility.weylEqNablaQ` | `7dda1ab2b8` |
| Corollary 3.5 | Integral clearing of denominators for dual Weyl elements | `Formalism.GradedWeylDivisibility.weylRational` | `3c24478050` |
| Lemma 4.1 | Adjoint stability along an $m$-step iterated chain | `Formalism.ThetaLiftIndependence.ChainScalarLemmas.mem` | `8f9405f30b` |
| Theorem 4.2 | Lattice-lift sums do not depend on class representatives | `Formalism.ThetaLiftIndependence.latLiftIndep` | `e7099e635a` |
| Theorem 4.3 | Duality and inversion of finite-field Gauss operators | `Formalism.ThetaLiftIndependence.final_gaussOp` | `b3a9090f39` |
| Theorem 4.4 | Leading principal minors on the big cell | `Formalism.QuantumDeterminants.bigCellMinors` | `88c03e52dd` |
| Theorem 4.5 | Harish-Chandra formal degree formula | `Formalism.QuantumDeterminants.fdegDimV` | `b0860544d9` |
| Theorem 5.1 | Boundedness of slopes and upper bound by $\mu_{\max}$ | `Formalism.F.slopesBdd` · `Formalism.F.slopeLeMuMax` | `67df59a53c` `5b89233162` |
| Theorem 5.2 | Minkowski short-vector theorem | `Formalism.F.shortVector` | `d6a1c982fb` |
| Theorem 5.3 | Additivity of rank and degree for saturated sublattices | `Formalism.F.deg_eq_deg_add_deg_quot` | `d5a6b0e8ae` |
| Theorem 5.4 | Minimal nonzero vector bounds the maximal slope | `Formalism.F.muMaxLeMin` | `30137ff06c` |
| Theorem 6.1 | Deterministic subset selection under binomial variance | `Formalism.BehrendSets.Final.select` | `2b17c057fd` |
| Theorem 6.2 | Triviality of large prime-power subgroups in $\mathrm{GL}_k(E)$ | `Formalism.BehrendSets.FinalE.trivStab` | `23ff82c789` |

| Item | Title | Lean declaration | Signature digest |
|---|---|---|---|
| Theorem 6.3 | Existence of local Langlands correspondence with type law | `UpperTheory.Zero.LLC.recTExists` | `79e511e45e` |
| Theorem 6.4 | Strong local-global Galois integral structure from theta lifts | `UpperTheory.Gal.Int.galoisLocalGlobalStrong_recParameter` | `6dbdb89969` |
| Theorem 6.5 | Global Selmer-length bound across primes | `UpperTheory.V.SmallClassNumbers.corGK` | `00c2c25c4c` |
| Theorem 6.6 | Dense threshold connectivity implies primitive right-half-plane nonvanishing | `UpperTheory.Top.zeroFreeAll_of_topG` | `44ddf8874b` |
| Theorem 6.7 | Assembly of right-half-plane nonvanishing for all Dirichlet characters | `UpperTheory.Top.plugAll` | `21a7dc8ac2` |
| Theorem 6.8 | Unconditional right-half-plane nonvanishing for all Dirichlet $L$-functions | `UpperTheory.FinalTheorems.everyCharacter` | `f0305d3527` |
| Theorem 7.1 | Kernel bundle: nonvanishing for every Dirichlet character | `allChi` | `fe66c5e18d` |
| Theorem 7.2 | Kernel bundle: the Riemann Hypothesis | `rhOfAll` | `11ab6e98f3` |
| Theorem 7.3 | Kernel bundle: critical line for primitive characters | `critAll` | `c73d13e98a` |
| Theorem 7.4 | Kernel bundle: half-plane, strip, and zeta conjunction | `claimAll` | `4df9f05887` |

## Written proofs

These results cite a DOI or source repository, book proofs and registered formalization requests. They are not kernel-verified and are excluded from LEAN √ counts.

(none)

## Open problems and conjectures

The following items are explicitly marked open; the book does not treat them as proved.

- Conjecture Coda.1 — Extended Riemann Hypothesis (ERH) for general Dedekind zeta functions
- Conjecture Coda.2 — Grand Riemann Hypothesis for general automorphic $L$-functions
- Conjecture Coda.3 — Simplicity of zeta zeros
- Open problem Coda.4 — Montgomery's pair correlation and GUE spectral statistics